THE MATHEMATICAL GAZETTE

EDITED BY
T. A. A. BROADBENT, M.A.
WILLS HALL, BRISTOL, 9.

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CONTENTS.

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THE MATHEMATICAL ASSOCIATION.

(An Association of Teachers and Students of Elementary Mathematics.)

"I hold every man a debtor to his profession; from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves, by way of amends, to be a help and an ornament thereunto."—BACON (Preface, *Maxims of Law*).

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CHARLES PENDLEBURY, 1854-1941.

I.

In an old number of the *Cornhill* there is an article in which the habits and character of C. Pendlebury are deduced from the examples in the famous *Arithmetic*. This number was among Pendlebury's treasured possessions and perhaps few best-sellers have evoked so delicate a compliment, for it is not always easy to remember that behind the overwhelming figures of millions of sales there is some real human being. Nor is it always easy to remember that behind the steady growth of our Association, from the few sturdy heretics who met under J. M. Wilson's chairmanship at Rugby in 1871 to the 1700 members of to-day, there has been the driving force of some outstanding personalities. Pendlebury joined the Association in 1885, became a Secretary in 1886 and held this office till his resignation in 1936. During that fifty years, the Association dethroned Euclid and reformed the teaching of geometry, threw out Branches in Great Britain and Australia, published a long series of authoritative reports on the teaching of mathematics and a journal which is now not far from its three-hundredth number. In all these activities, as well as in the routine of his official duties, Pendlebury played a part, if often a part in the background. He abhorred that efficiency which cannot proceed without ostentation. But if you could persuade him that some course of action should be taken, you might safely leave it to him to carry out. You would go to him with a suggestion for some change or other ; he would listen to you with the courteous care of the deaf (in later years he was very deaf), and always with that benevolent smile which seemed to say : " Dear me, these young folk are really growing up " ; he would point out

the weak places in your plan, perhaps even tell you how such a scheme had been tried and had failed thirty years ago. But in a short time you would discover that the good points in your idea had been quietly and efficiently put into operation.

Pendlebury filled a long life with many activities. In this place it is particularly our duty and pride to record the long connection between our Association and one who, both as teacher and as writer, exercised so powerful an influence on the teaching of our subject.

T. A. A. B.

II.

My personal acquaintance with Charles Pendlebury began some thirty years ago when I joined the Council of the Mathematical Association. The meetings were then held in the Staff Room of my own college in Southampton Row, around what was normally our tea-table. Pendlebury sat at the President's elbow with his colleague (and my colleague) Miss Punnett by his side—the ideal secretary, quiet and unobtrusive, but keeping a sharp eye upon whatever was proposed or transacted, and omniscient about the Association's business affairs. These were close, lively and multiplex associations, yet somehow they did not enable me to identify the secretary of the Mathematical Association with the author of Pendlebury's *Arithmetic*; these two conceptions maintained in my mind a distinctness which was, I think, never wholly overcome.

As rumination over old times awakens this strange suspicion, I seem to understand, better than before, the success of the famous book. The elements that go to the making of some books are so intimately fused that one can hardly think of them as separable; they compose entities endowed with a unity that defies bifurcation into a writer and something he has written. For most people Boswell's *Johnson* is one of these, and, for thousands of boys and girls at school—perhaps even for their teachers—Pendlebury's *Arithmetic* must have been another. It never occurred to them that a man once sat down before a pile of blank paper and began slowly to make the book; for them it was just one of the wonders of creation that one accepts without inquiry about its genesis.

When this kind of thing happens it must be because the author puts so much of himself into his book that it acquires an active personality of its own, reflecting the author's, yet in a queer way independent of it. This miracle is, I think, visible in Pendlebury's *Arithmetic*; for its unromantic pages express in a curiously living way the lineaments of his clear and orderly mind, his gentle firmness of purpose, his quiet steadiness of character.

But when Pendlebury started out to make his book he was undoubtedly favoured by the *Zeitgeist*. In the mid-Victorian years Isaac Todhunter had created a series of mathematical textbooks which are, one fears, remembered now only by old-stagers, but ought to be reckoned among the soundest achievements of English education. He dealt memorably with Euclid, algebra and trigonometry,

to say nothing of higher studies, but, except for a work on mensuration, left arithmetic alone. The old-stagers among us must often have gained their knowledge of arithmetic (as I did) from a textbook written, probably, by an itinerant teacher of its mysteries, one who was in a position to impart valuable tips to many kinds of commercial and professional people—" mixtures " for the grocer, " tare and tret " for the transport clerk, " duodecimals " for the architect and builder, rules for a timber merchant to use in sizing up timber, and so forth. But towards the end of the reign the propaganda of the Society for the Improvement of Geometrical Teaching and other innovating influences had made that kind of thing obsolete ; mathematics had won a well-assured place in the secondary school curriculum, and it was seen that arithmetic, freed from its compromising connections and treated scientifically, should be taught as an essential part of it. Notwithstanding meritorious efforts which need not be mentioned here, the " Todhunter " in arithmetic had not yet emerged. It was to appear in 1886 under Pendlebury's name—a work endowed with all the Todhunter virtues, but with what was then a refreshing modernity of treatment, and with experience of the needs and capabilities of boys embodied in every page.

Tout passe, and Pendlebury's work may some day be superseded. But if that day comes to it, as it came to Todhunter's, its successors will for many generations owe a great and, one hopes, not unacknowledged debt to one of the most honest, capable and thorough textbooks of an earlier time.

T. P. Nunn.

III.

In trying to give an impression of the work done by Mr. Pendlebury in his fifty years as honorary secretary of the Mathematical Association, I find myself seriously handicapped by the fact that, during the earlier and most important part of that work, I had no contacts with him or with the Association. During my many years of collaboration with him, however, I came to know him well enough to be able to form a fairly vivid idea of his activities during those earlier years, when the Association was still young and some of its most characteristic features were in process of development. I can picture him, quiet, imperturbable, efficient, always ready to produce from the unfailing storehouse of his memory helpful facts and figures, together with wise comments and suggestions and timely warnings. And there was nothing colourless about his contributions to discussion ; everyone who ever worked with him knows how deeply he had the interests of the Association at heart, and what definite views he held as to its proper position and functions. He took a keenly interested part in the first formation and subsequent development of the Teaching Committees and kept careful and conscientious watch over the carrying out of the details of their constitution. Throughout the whole of his long period of office he was untiring in his efforts to take every opportunity of founding new Branches,

and, in general, of increasing the membership of the Association. It was, I remember, a red-letter day for him on which the number of members reached a thousand. He was, for once, really excited.

It was his unfailing attention to the details of the business of the Association that led to that smooth running of its affairs that everyone learned to take for granted, often, perhaps, without fully realising the efficiency that was behind it. In this connection it may be worth recalling that for a large part of his period of office the whole work of arranging the programme of the Annual Meeting was in his hands. It was only when the meeting overflowed into two days, at a time when the increased membership added greatly to his other work, that he felt the task too much for him and asked to have a programme committee appointed to take that business off his hands. I must confess that when I first became his fellow secretary, it was with much trepidation lest I should prove wanting and the efficiency of the Association's " secretariat " should be lowered. But his steady support and kindly, good-humoured reminders made things easy: " Isn't it about time the notices about " so-and-so " went out?" " Don't forget that " such-and-such a report " should be in the printers' hands " by such a date, and so on.

Mr. Pendlebury was for so long an integral part of the Association that for many people it was hard to imagine it without his guidance or to realise the fact of his resignation when it came. And now his death has brought to those who knew him best an added sense of personal loss. It is sad to know that never again shall we see him, even as an unofficial member, at our meetings or exchange views with him as to " how things are going ". MARGARET PUNNETT.

GLEANINGS FAR AND NEAR.

1385. The four Indians sat for some time watching how the strangers roasted their meat and cooked their rice.

" You've come a long way, no doubt ", one of them said at last, " and no doubt you have a long way still to go. You are very clever men, that's sure."

" We can read books," Curtin replied, " and we can write letters, and we can reckon with figures."

" With figures? " another of the four asked. " Figures? We don't know that."

" Ten is a figure," Curtin explained. " And five is a figure."

" Oh," said another of their guests. " That's only half of it. Ten is nothing, and five is nothing. You mean ten fingers or five beans or three hens. Is that it? "

" That's so," Howard put in.

The Indians laughed, because they had understood, and one of them said: " You can't say ten. You must always say ten what. Ten trees or ten men or ten birds. If you say ten or five or three without saying what you mean, there's a hole and it's empty."—B. Traven, *The Treasure of the Sierra Madre*. [Per Dr. D. Pedoe.]

AN ACCOUNT OF 4-PIECE MECHANISMS IN THREE DIMENSIONS.

By R. H. Macmillan.

The following paper is an attempt to explain the action of a class of three-dimensional mechanisms, first from a theoretical standpoint and afterwards to suggest some practical applications. One or two rather more general mechanisms are investigated than have previously been studied and well-known mechanisms are deduced as special cases of these. There is no modern textbook or recent publication dealing with these mechanisms and their properties are not generally known.

Definitions.

Any member, A, of a mechanism is called *a link*. If its motion, relative to another link, B, is that of pure turning about some axis, fixed relative to B, this axis is called the common hinge-line of A and B. (B's motion, relative to A, is also one of pure turning about the same hinge-line, fixed relative to A.) For example, the hinge-line of the page of a book (considered rigid) and the table it rests upon is the back of the book ; the hinge-lines of the femur of a man riding a bicycle are the horizontal lines, perpendicular to the frame of the bicycle, passing through his knee and the saddle of the bicycle. If, for the sake of the argument, his ankle is assumed to be rigid, then the two joints of the leg, the pedal crank and the frame of the bicycle form a closed two-dimensional 4-piece chain, which turns oscillatory into rotary motion.

More rigidly, then :

A hinge-line is the common axis about which two adjacent links of a mechanism turn.

A turning-pair consists of two adjacent links and their common hinge-line.

The length of a link is the perpendicular distance between its hinge-lines.

The twist of a link is the angle between its hinge-lines.

A closed chain of n turning-pairs or *an n-piece chain* consists of n links and their n hinge-lines arranged so that the perpendiculars from any hinge-line, D, to its adjacent hinge-lines, E and C, at the further ends of the two links connected by D, are *concurrent on* D.

A Mechanism is formed from a chain if one link is fixed and the others are capable of a definite relative motion.* If this link is freed and another link fixed, the mechanism formed is called *an inversion* of the first.

(*Pure sliding motion* is produced by the turning of two links upon one another when their common hinge-line has receded to an infinite distance ; but in this paper it does not concern us.)

These definitions are merely an extension of the Reuleaux System to three dimensions.

* (*I.e.*, have one degree of freedom.)

It is often convenient to visualise a link as the straight line perpendicular to its two hinge-lines ; when these are concurrent, however, the link may be visualised as consisting of the plane of the two hinge-lines (which cut at some fixed angle).

LEMMAS.

Before proceeding to a discussion of the various mechanisms, it will be convenient to prove one or two relationships that will frequently be useful later.

A. If $$\tan\theta \, . \, \tan\phi = k$$

$$\log\tan\theta + \log\tan\phi = \log k.$$

Differentiating, $$\frac{2}{\sin 2\theta} + \frac{2}{\sin 2\phi} \cdot \frac{d\phi}{d\theta} = 0.$$

Hence $$-\frac{d\phi}{d\theta} = \frac{\sin 2\phi}{\sin 2\theta}. \qquad \text{......(1)}$$

Again, $$-\frac{d\phi}{d\theta} = \frac{\left(\dfrac{2\tan\phi}{1+\tan^2\phi}\right)}{2\sin\theta\cos\theta} = \frac{\left(\dfrac{k/\tan\theta}{1+k^2/\tan^2\theta}\right)}{\sin\theta\cos\theta},$$

and so $$-\frac{d\phi}{d\theta} = \frac{k}{\sin^2\theta + k^2\cos^2\theta}. \qquad \text{......(2)}$$

B. If $$\cos\phi = (b - a\cos\theta)/(a - b\cos\theta),$$

then

$$\sin\phi = (a^2 - 2ab\cos\theta + b^2\cos^2\theta - b^2 + 2ab\cos\theta - a^2\cos^2\theta)^{\frac{1}{2}}/(a - b\cos\theta)$$
$$= \sqrt{(a^2 - b^2)}\sin\theta/(a - b\cos\theta),$$

and $$a - b\cos\phi = (a^2 - b^2)/(a - b\cos\theta),$$

or $$(a - b\cos\theta)(a - b\cos\phi) = a^2 - b^2.$$

Differentiating

$$(a - b\cos\phi)\, b\sin\theta \, . \, d\theta + (a - b\cos\theta)\, b\sin\phi \, . \, d\phi = 0,$$

and so $$-\frac{d\phi}{d\theta} = \frac{\sin\theta(a - b\cos\phi)}{\sin\phi(a - b\cos\theta)}.$$

Thus $$-\frac{d\phi}{d\theta} = \frac{a - b\cos\phi}{\sqrt{(a^2 - b^2)}} = \frac{\sqrt{(a^2 - b^2)}}{a - b\cos\theta}. \qquad \text{......(3)}$$

Similarly, if $$\cos\phi = (b + a\cos\theta)/(a + b\cos\theta),$$

then $$\frac{d\phi}{d\theta} = \frac{\sqrt{(a^2 - b^2)}}{a + b\cos\theta}. \qquad \text{......(4)}$$

It can also be shown that when

$$\cos\phi = (b - a\cos\theta)/(a - b\cos\theta), \qquad \text{......(B)}$$

then $$\tan\tfrac{1}{2}\theta\tan\tfrac{1}{2}\phi = \sqrt{\{(a+b)/(a-b)\}}, \qquad \text{......(A)}$$

so that expressions *A* and *B* are interconnected.

4-Piece Mechanisms.

The 7-piece chain has, in general, one degree of freedom (*i.e.*, when one link is fixed, it will, in general, behave as a mechanism). The general 6-piece has, therefore, *one* degree of restraint (*i.e.*, it will be just rigid) and, in general, the 4-piece will have *three degrees of restraint*. It will be realised, therefore, that it is altogether exceptional for a 4-piece to behave as a mechanism.

There are two special forms of 4-piece known in which this does occur :

(1) *The " Spheric " chains*, in which the four hinge-lines meet in a point, called the " pole " of the chain (and hence the lengths of all the links are zero) ; and

(2) *The " Skew Isogram "*, in which the lengths of alternate links are equal and their twists must obey a specified equation.

The Spheric Chains.

These are sometimes called " conic " chains, but the name is not to be recommended as it is misleading and is not suggestive of the best method of studying their behaviour ; they are discussed in Kennedy's *The Mechanics of Machines*, pp. 488-541.

The chain is most easily pictured as the relative motion of four planes AOB, BOC, COD and DOA, where O is the pole. (It is very

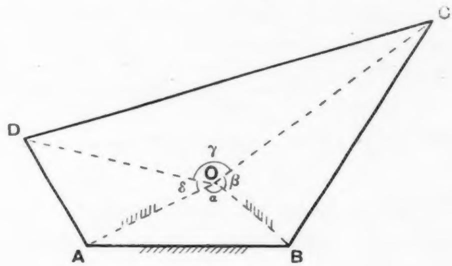


Fig. 1.

easy to make paper models with any required angles between the hinge-lines.)

To define the motion of the mechanism it is necessary to fix one link ; let this be AB. This fixes the plane AOB ; let this plane be horizontal. Let the planes AOD, BOC make angles θ and ϕ with the vertical respectively, and let the angles between the hinge-lines ($\angle AOB$, $\angle BOC$, $\angle COD$ and $\angle DOA$) be α, β, γ and δ. If these angles are given definite values, the mechanism is uniquely defined. Then if AOD has, at any instant, angular velocity $\omega = d\theta/dt$ and BOC has angular velocity $\omega' = d\phi/dt$, then $\omega'/\omega = d\phi/d\theta$ and this is called the *Velocity Ratio* of the chain.

It is required first to find the relation between θ and ϕ in terms of the known angles : it will be noticed that we are at liberty to

make OA, OB, OC and OD any lengths we may choose, so that if AD and BC are chosen perpendicular to OA and OB, then θ and ϕ are the angles which AD and BC make with the vertical.

The problem can be tackled by straightforward orthogonal projection with the following construction :

C', D' are the projections of C, D on to plane AOB, which is fixed and horizontal.

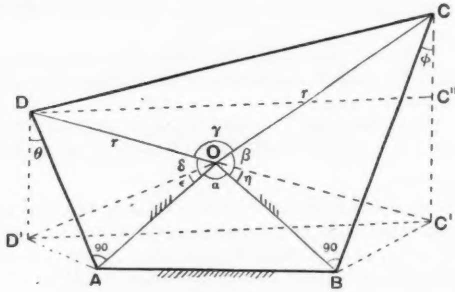


FIG. 2.

Draw DC'' parallel to $D'C'$ to meet CC' in C''.

Let $AOD' = \epsilon$, $BOC' = \eta$, and let $OD = OC = r$.

We now use the following facts :

(a) $DC^2 = CC''^2 + DC''^2$

$$= D'C'^2 + (CC' - DD')^2.$$

(b) $D'C'^2 = OD'^2 + OC'^2 - 2OD' \,.\, OC' \,.\, \cos(\alpha + \epsilon + \eta)$,

and also

$$DD' = r \sin \delta \cos \theta, \quad CC' = r \sin \beta \cos \phi\ ;$$

$$OD' = r\sqrt{(1 - \sin^2 \delta \cos^2 \theta)}, \quad OC' = r\sqrt{(1 - \sin^2 \beta \cos^2 \phi)}\ ;$$

$$DC = 2r \sin \tfrac{1}{2}\gamma.$$

From these we obtain :

$$(\sin^2 \delta \cos^2 \theta + \sin^2 \beta \cos^2 \phi) + 2[\sqrt{\{(1 - \sin^2 \delta \cos^2 \theta)(1 - \sin^2 \beta \cos^2 \phi)\}} \cos(\alpha + \epsilon + \eta) + \cos \gamma] = (\sin \delta \cos \theta - \sin \beta \cos \phi)^2, \quad \ldots\ldots\ldots\ldots(5)$$

which is extremely unwieldy when we substitute for ϵ and η in terms of known angles. In particular cases the expression may be less complicated, though the algebra is always heavy.

The method is good when $\beta = \delta = 90°$ and consequently $\epsilon = \eta = 90°$. The relation between θ and ϕ is then :

$$\cos \theta \cos \phi - \sin \theta \sin \phi \,.\, \cos \alpha + \cos \gamma = 0,$$

which reduces to Hooke's joint (*inf.*) when $\gamma = 90°$, $\cos \gamma = 0$.

A far better and more logical method of attacking the problem is by spherical trigonometry. Those formulae which are of value to us are that, in any spherical triangle ABC,

I. $\cos a = \cos b \cos c + \sin b \sin c \,.\, \cos A$. (p. 21.)

II. $\dfrac{\sin a}{\sin A} = \dfrac{\sin b}{\sin B} = \dfrac{\sin c}{\sin C}$. (p. 25.)

III. $\tan \frac{1}{2}(A+B) = \dfrac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \dfrac{C}{2}$. (p. 34.)

IV. $\dfrac{\sin (A+B)}{\sin C} = \dfrac{\cos a + \cos b}{1 + \cos c}$. (p. 45.)

Further, in any triangle ABC, having $\angle C = 90°$,

V. $\cos A = \tan b / \tan c$. (p. 48.)

VI. $\tan A \,.\, \tan B = 1/\cos c$. (p. 48.)

These are to be found in any standard textbook of spherical trigonometry, but the references are to *Spherical Trigonometry*, by I. Todhunter and J. G. Leathem, 1925 ed.

Consider the spheric chain with its pole at the centre of the unit sphere ; the intersections of the four planes with the surface will be great circle arcs and will form a " spherical quadrilateral ", with AB, BC, CD and DA equal to α, β, γ and δ.

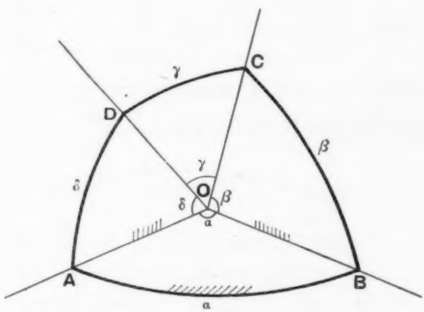


FIG. 3.

Fix AOB horizontally and let AOD, BOC make angles θ and ϕ with the vertical,

then $$\begin{cases} DAB = (90 + \theta), \\ ABC = (90 + \phi). \end{cases}$$

Consider now the particular case in which $\alpha=\beta=\gamma=\delta$; $ABCD$ is a "spherical rhombus". (Fig. 4). Join AC, BD by great circle arcs and let them intersect in X. $\angle AXB=90°$.

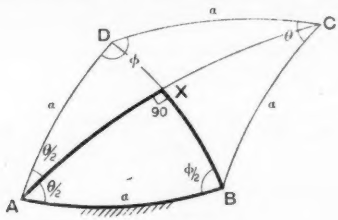


FIG. 4.

Let AD, BC make angles θ and ϕ with the fixed plane AOB. Applying VI to $\triangle AXB$,

$$\tan\frac{\theta}{2}\tan\frac{\phi}{2}=\frac{1}{\cos\alpha}=\text{constant.}$$

Thus from relation (2), $-\dfrac{d\phi}{d\theta}=\dfrac{\cos\alpha}{\cos^2\frac{1}{2}\theta+\cos^2\alpha\sin^2\frac{1}{2}\theta}$,

which reduces to $$-\frac{d\phi}{d\theta}=\frac{\cos\alpha}{1-\sin^2\alpha\,.\,\sin^2\frac{1}{2}\theta}.$$

The velocity ratio fluctuates between $1/\cos\alpha$ and $\cos\alpha$.

When $\alpha=30°$, $\sin\alpha=\frac{1}{2}$, $\cos\alpha=\frac{1}{2}\sqrt{3}$, $\omega'=\dfrac{-4\sqrt{3}\omega}{7+\cos\theta}$.

$\alpha=45°$, $\sin\alpha=\cos\alpha=1/\sqrt{2}$, $\omega'=\dfrac{-2\sqrt{2}\omega}{3+\cos\theta}$.

$\alpha=60°$, $\sin\alpha=\frac{1}{2}\sqrt{3}$, $\cos\alpha=\frac{1}{2}$, $\omega'=\dfrac{-4\omega}{5+3\cos\theta}$.

Consider also the case in which $\gamma=\alpha$, $\delta=\beta$.

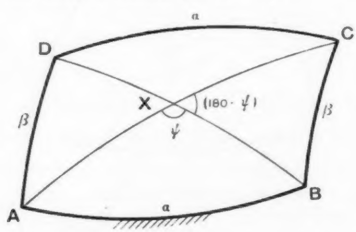


FIG. 5.

Join AC, BD by great circle arcs; then since $ABCD$ is a "spherical parallelogram" these will *bisect* each other in X.

Let $\angle AXB=\psi$, $BD=2x$, $AC=2y$.

From the triangles AXB, BXC:

$$\left.\begin{aligned}\cos\alpha &= \cos x\cos y + \sin x\sin y\,.\cos\psi\\ \cos\beta &= \cos x\cos y - \sin x\sin y\,.\cos\psi\end{aligned}\right\}, \text{ .... from I.}$$

Adding, $$\cos\alpha + \cos\beta = 2\cos x\,.\cos y. \quad\text{.......................(6)}$$

From the triangles ADB, ACB:

$$\left.\begin{aligned}\cos 2x &= \cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\theta = 2\cos^2 x - 1\\ \cos 2y &= \cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\phi = 2\cos^2 y - 1\end{aligned}\right\}. \quad\text{......(7)}$$

Eliminating x and y from (6) and (7):

$$(\cos\alpha + \cos\beta)^2 = (1 + \cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\theta) \times (1 + \cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\psi),$$

which is the required connection between θ and ϕ.

Now, let $$\mu = \tan^{-1}\left(\frac{\sin\alpha\sin\beta}{\cos\alpha + \cos\beta}\right),$$

then $$\sec\mu = \frac{1 + \cos\alpha\,.\cos\beta}{\cos\alpha + \cos\beta}.$$

Dividing by $(\cos\alpha + \cos\beta)^2$, we get:

$$1 = (\sec\mu + \tan\mu\cos\theta)(\sec\mu + \tan\mu\cos\phi).$$

Hence

$$\cos\phi = \frac{1 - \sec^2\mu - \tan\mu\sec\mu\cos\theta}{\tan\mu\sec\mu + \tan^2\mu\cos\theta} = -\frac{\tan\mu + \sec\mu\cos\theta}{\sec\mu + \tan\mu\cos\theta}.$$

Thus from (4), $$-\frac{d\phi}{d\theta} = \frac{\sqrt{(\sec^2\mu - \tan^2\mu)}}{\sec\mu + \tan\mu\cos\theta};$$

therefore $$-d\theta/d\phi = \sec\mu + \tan\mu\cos\theta.$$

Thus the velocity ratio fluctuates between $(\sec\mu \pm \tan\mu)$.

In full, $$-\frac{d\theta}{d\phi} = \frac{1 + \cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\theta}{\cos\alpha + \cos\beta}.$$

When $\alpha = 90$, $$-\frac{d\theta}{d\phi} = \frac{1 + \sin\beta\cos\theta}{\cos\beta}, \quad (\mu = \beta.)$$

When $\alpha = 90°$, $\beta = 30°$: $-d\theta/d\phi = (2 + \cos\theta)/\sqrt{3}$.

$\alpha = 90°$, $\beta = 45°$: $-d\theta/d\phi = \sqrt{2} + \cos\theta$.

$\alpha = 90°$, $\beta = 60°$: $-d\theta/d\phi = 2 + \sqrt{3}\cos\theta$.

When $\alpha = \beta$, $$\begin{aligned}-2\cos\alpha(d\theta/d\phi) &= 1 + \cos^2\alpha + \sin^2\alpha\cos^2\theta\\ &= 2 - \sin^2\alpha - \sin^2\alpha(2\sin^2\tfrac{1}{2}\theta - 1),\end{aligned}$$

and $$-\cos\alpha(d\theta/d\phi) = 1 - \sin^2\alpha\sin^2\tfrac{1}{2}\theta,$$

which is the form in which we had it previously.

When $\alpha = 30°$, $\beta = 60°$, $-d\theta/d\phi = \{2 + \tfrac{1}{2}\sqrt{3}\,.\,(1 + \cos\theta)\}/(1 + \sqrt{3})$.

Hooke's Joint and its Inversions.

These are the spheric chains in which

$$\beta = \gamma = \delta = 90°.$$

Let COD be horizontal; then if AC and BD cut in X it can at once be seen from the figure that X lies on the vertical through O. Thus $\angle AXB = 90°$.

Fix CDO, and let BOC, AOD make angles θ and ϕ with the vertical; then $BX = \theta$, $AX = \phi$.

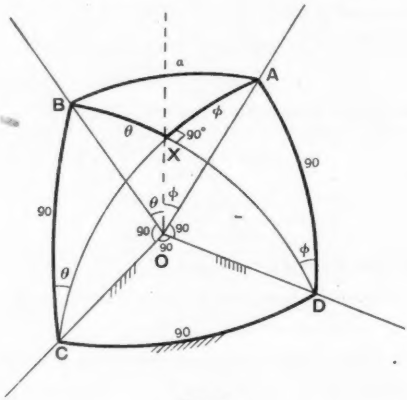


Fig. 6.

In the triangle ABX,

from I ... $\cos\alpha = \cos\theta\cos\phi$.

Thus $$\frac{d\phi}{d\theta} = -\frac{\tan\theta\cos\alpha}{\sqrt{(\cos^2\theta - \cos^2\alpha)}} \text{ and } \theta < \alpha.$$

When $$\theta = \alpha, \quad d\theta/d\phi = 0;$$
$$\theta = 0, \quad d\phi/d\theta = 0.$$

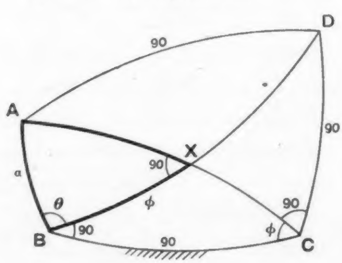


Fig. 7.

Now *fix link BC*; let BOC be horizontal, then if BAO, CDO make angles θ and ϕ with the vertical, $BCX = \phi = BX$, $ABX = \theta$. (Fig. 7.)

In the triangle ABX,

$$\text{from V} \ldots \cos\theta = \tan\phi / \tan\alpha.$$

Hence when $\theta = 0$ and $180°$, $\phi = +\alpha$ and $-\alpha$.

If *plane AOB is fixed* we have Hooke's joint.

Let AOB be horizontal and let BOC make an angle ϕ *with the vertical*; let AOD make an angle θ *with the horizontal*, this being the usual convention.

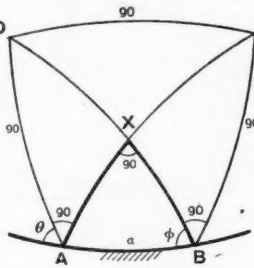


FIG. 8.

In the triangle ABX,

$$\text{from VI} \ldots \tan\phi \tan(90 - \theta) = 1/\cos\alpha.$$

Hence $$\tan\theta = \tan\phi\cos\alpha.$$

This mechanism is discussed more fully later.

THE GENERAL SPHERIC CHAIN.

The relation between θ and ϕ in the general spheric chain has been given by Dr. G. T. Bennett of Emmanuel College, Cambridge, as:

$$\begin{aligned}\cos\gamma = \cos\delta\cos\alpha\cos\beta - \sin\delta\cos\alpha\sin\beta \,.\, \cos\theta\cos\phi \\ + \sin\delta\sin\alpha\cos\beta \,.\, \cos\theta + \cos\delta\sin\alpha\sin\beta \,.\, \cos\phi \\ + \sin\beta\sin\delta \,.\, \sin\theta\sin\phi.\end{aligned}$$

Dr. Bennett proves this by constructing a quadrantal triangle ($a = b = c = 90°$), AYZ, upon AB produced and using the fact:

$$\cos CD = \cos CA \,.\, \cos DA + \cos CY \,.\, \cos DY + \cos CZ \,.\, \cos DZ.$$

The following proof, however, is straightforward and symmetrical:

Use the abbreviated notation:

$$DY \equiv \cos DY, \quad DY' \equiv \sin DY.$$

Draw DY and CZ perpendicular to AB to meet at X. (Fig. 9.)

$$XY = XZ = 90° ; \quad \angle YXZ \rightarrow YZ.$$

In the right-angled triangles ADY, BCZ :

$$DY' = \delta'\theta' ; \quad CZ' = \beta'\phi' \quad \text{from II.}$$

$$\theta = \frac{\tan AY}{\tan \delta} = \frac{AY'}{AY} \cdot \frac{\delta}{\delta'} ; \quad \phi = \frac{\tan BZ}{\tan \beta} = \frac{BZ'}{BZ} \cdot \frac{\beta}{\beta'} \quad \text{from V.}$$

$$\delta = AY \,.\, DY ; \quad \beta = BZ \,.\, CZ \quad \text{from I.}$$

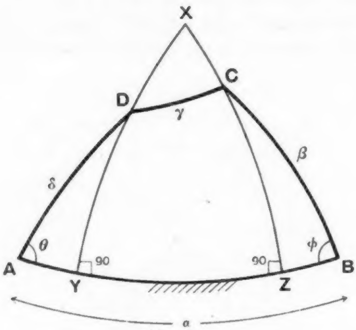


FIG. 9.

Thus $$DY = \frac{\delta}{AY} = \frac{\delta'\theta}{AY'} \quad \text{and} \quad CZ = \frac{\beta}{BZ} = \frac{\beta'\phi}{BZ'}.$$

Now, from the triangle CDX,

$$CD = CX' \,.\, DX'(DXC) + CX \,.\, DX \quad \text{from I.}$$

$$= DY \,.\, CZ \,.\, YZ + DY' \,.\, CZ'.$$

But $$YZ = \cos\{\alpha - (AY + BZ)\}$$

$$= \cos\alpha \cos(AY + BZ) + \sin\alpha \sin(AY + BZ).$$

Hence $$CD = DY \,.\, CZ\{\alpha \,.\, AY \,.\, BZ - \alpha \,.\, AY' \,.\, BZ' + \alpha' \,.\, AY' \,.\, BZ + \alpha' \,.\, AY \,.\, BZ'\} + DY' \,.\, CZ'.$$

Thus $$\gamma = \alpha \,.\, \delta \,.\, \beta - \alpha \,.\, \delta'\theta \,.\, \beta'\phi + \alpha' \,.\, \delta'\theta \,.\, \beta + \alpha' \,.\, \delta \,.\, \beta'\phi + \delta'\theta' \,.\, \beta'\phi'.$$

Q.E.D.

It will be found that if we put $\alpha = \gamma$, $\beta = \delta$ in this expression the result will factorise. This is due to the fact that in all spheric chains in which $\alpha \pm \beta = \gamma \pm \delta$ there will be an ambiguity of motion at the zero positions, and in all these cases it should be possible to factorise the expression obtained from substituting the known values of α, β, γ, and δ and each factor will account for a possible mode of motion. Whether the form of connection between θ and ϕ, given by substituting in this general equation, will give us $d\theta/d\phi$ is by no means certain and it is often quicker to work from first principles on an individual case than to attempt to transform, suitably, the equation obtained from the general expression.

THE SKEW ISOGRAM.

This mechanism was discovered and named by Dr. Bennett (*sup.*) and was first described by him in *Engineering*, 4/12/1903, p, 777; further papers by him, relating to it, are in *Engineering*, 19/12/1904, and *London Mathematical Society's Proceedings*, Vol. 13, p. 151. There is a brief reference to the mechanism in Bricard's *Cinématiques et Méchanismes*, pp. 159-161.

The problem may be treated by projection :

Let the lengths of the links AB, CD be a, and BC, DA be b.

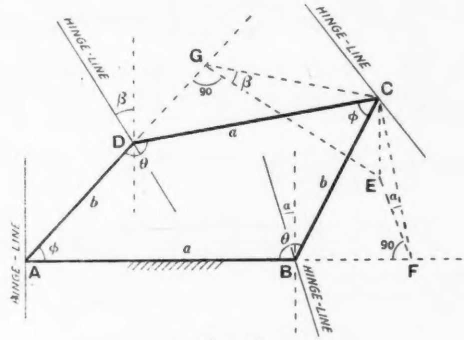


FIG. 10.

Fix AB with its hinge-line at A vertical, so that the plane ABD is horizontal.

Let E be the projection of C on to plane ABD. Draw CG and CF perpendicular to DA and AB.

Let the twist of AB be α; this is the angle between planes BAD, $ABC = \angle CFE = \alpha$.

Similarly, if twist of $DA = \beta$, $\angle CGE = \beta$.

If $\angle ABC = \theta$, $\angle DAB = \phi$; then, by joining AC and BD we see from congruent triangles $\angle CDA = \theta$, $\angle BCD = \phi$.

Now, $$CE = CF \sin\alpha = b\sin(\pi - \theta)\sin\alpha,$$

and $$CE = CG \sin\beta = a\sin(\pi - \theta)\sin\beta.$$

Thus $$a\sin\beta = b\sin\alpha, \quad \text{.............................(8)}$$

and this is the equation which the twists of opposite links must satisfy.

Project AGE on to AF :

$$AF = AG\cos\phi + GE\sin\phi.$$

Then

$$a + b\cos(\pi - \theta) = \{b + a\cos(\pi - \theta)\}\cos\phi + CG\cos\beta\,.\,\sin\phi$$

$$a - b\cos\theta = (b - a\cos\theta)\cos\phi + a\sin(\pi - \theta)\cos\beta\,.\,\sin\phi. \quad \text{...(9)}$$

Similarly, by projecting AFE on to AG, we get,

$$b - a\cos\theta = (a - b\cos\theta)\cos\phi + b\sin\theta \,.\, \cos\alpha \,.\, \sin\phi. \quad \text{.......(10)}$$

Adding (9) and (10),

$$(a+b)(1-\cos\theta)(1-\cos\phi) = \sin\theta\sin\phi(a\cos\beta + b\cos\alpha).$$

Thence $\tan\frac{1}{2}\theta\tan\frac{1}{2}\phi = (\sin\alpha\cos\beta + \cos\alpha\sin\beta)/(\sin\alpha + \sin\beta)$,

using equation (8), $\quad = \sin(\alpha+\beta)/(\sin\alpha + \sin\beta)$.

Therefore $\quad \tan\frac{1}{2}\theta\tan\frac{1}{2}\phi = \dfrac{\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)} = \text{constant}. \quad$ ....(11)

Thus, from (1), $-d\phi/d\theta = \sin\phi/\sin\theta$, and since we have been able to find a velocity ratio, this is sufficient proof that the chain works as a mechanism and is not rigid.

$$\tan\tfrac{1}{2}\theta\tan\tfrac{1}{2}\phi = \frac{\cos\frac{1}{2}\alpha\cos\frac{1}{2}\beta - \sin\frac{1}{2}\alpha\sin\frac{1}{2}\beta}{\cos\frac{1}{2}\alpha\cos\frac{1}{2}\beta + \sin\frac{1}{2}\alpha\sin\frac{1}{2}\beta} = \frac{1 - \tan\frac{1}{2}\alpha\tan\frac{1}{2}\beta}{1 + \tan\frac{1}{2}\alpha\tan\frac{1}{2}\beta}.$$

Thus $\quad \tan\frac{1}{2}\alpha\tan\frac{1}{2}\beta = \dfrac{1 - \tan\frac{1}{2}\theta\tan\frac{1}{2}\phi}{1 + \tan\frac{1}{2}\theta\tan\frac{1}{2}\phi} = \dfrac{\cos\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)}$.

In an ellipse of semi-axes a and b, $b^2 = a^2(1-e^2)$, where e is the eccentricity. It is a property of the ellipse that

$$\tan\tfrac{1}{2}\theta\tan\tfrac{1}{2}\phi = (1-e)/(1+e),$$

where θ and ϕ are the angles, measured towards the centre of the ellipse, between the major axis and the focal radii to any point on the ellipse. If, then, we draw an ellipse with $e = \tan\frac{1}{2}\alpha\tan\frac{1}{2}\beta$, we have a construction for θ and ϕ, as related by equation (11).

We may also deduce from equations (9) and (10), by subtraction,

$$\frac{\cos\theta + \cos\phi}{\sin\theta\sin\phi} = \frac{\sin\alpha\sin\beta}{\cos\alpha + \cos\beta} = k = \tan\mu, \text{ say.} \quad \text{...........(12)}$$

From this expression we can obtain $d\theta/d\phi$ by a method precisely similar to that employed for the spheric chain in which $\alpha = \gamma$, $\beta = \delta$.

It will be of greater interest, however, to treat the problem of the skew isogram also by spherical trigonometry. Draw from the centre of the unit sphere lines parallel to the four hinge-lines and join their intersections with the surface by great circle arcs; these will form a "spherical parallelogram" with the angles θ and ϕ as shown in Fig. 11.

Join DB by a great circle arc. Then $\angle ADB = \angle DBC$.

Then, by applying III to the triangle ABD,

$$\tan\tfrac{1}{2}(ADB + DBA) = \frac{\cos\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}(\alpha+\beta)}\cot DAB.$$

i.e., $\quad \tan\frac{1}{2}\theta\tan\frac{1}{2}\phi = \cos\frac{1}{2}(\alpha+\beta)/\cos\frac{1}{2}(\alpha-\beta). \quad$ ....(11)

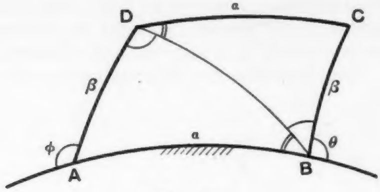


FIG. 11.

Further, from IV, $\dfrac{\sin(ADB+DBA)}{\sin DAB}=\dfrac{\cos\alpha+\cos\beta}{1+\cos DB}$.

Substituting for $\cos DB$, by I,

$$\frac{\sin\theta}{\sin\phi}=\frac{1+\{\cos\alpha\cos\beta+\sin\alpha\sin\beta\cos(180-\theta)\}}{\cos\alpha+\cos\beta}=-\frac{d\theta}{d\phi}.$$

Hence
$$-d\theta/d\phi=\sec\mu-\tan\mu\cos\theta$$
$$=\sqrt{(1+k^2)}-k\cos\theta,$$

where μ and k have the values of equation (12).

This method might, of course, be applied equally well to the spheric chain above, with $\alpha=\gamma$, $\beta=\delta$.

Consider the case in which $\alpha=90$; $\mu=\beta$, $b/a=\sin\beta$

$$-d\theta/d\phi=\sec\beta-\tan\beta\cos\theta$$
$$=(a-b\cos\theta)/\sqrt{(a^2-b^2)}\,;$$

$-\omega'$ fluctuates between $\omega\sqrt{\{(a+b)/(a-b)\}}$ and $\omega\sqrt{\{(a-b)/(a+b)\}}$.

This special case, and that in which $a=b$, are of sufficient interest to be worth proving from first principles by projection.

When $\alpha=90°$, $\sin\beta=b/a$ $(b<a)$; E now coincides with F in Fig. 10 (*see* Fig. 15, *inf.*). From the triangle AGE, in which $\angle AGE=90°$, we have $AE\cos\phi=AG$ and hence

$$\{a+b\cos(\pi-\theta)\}\cos\phi=b+a\cos(\pi-\theta).$$

Thus
$$\cos\phi=(b-a\cos\theta)/(a-b\cos\theta)$$

and so, from relation (3)

$$-\frac{d\phi}{d\theta}=\frac{\sqrt{(a^2-b^2)}}{a-b\cos\theta}=\frac{\sqrt{(a^2-b^2)}}{AE}.$$

When $a=b$, $\alpha=\beta$. Fig. 10 becomes symmetrical about AE. Projecting AGE on to AF,

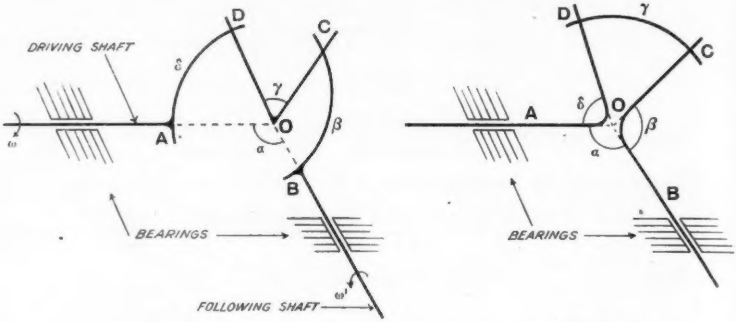
$$(a-a\cos\theta)\cos\phi+a\sin\theta\sin\phi\cos\alpha=a-a\cos\theta.$$

Thus
$$\cos\alpha=\frac{(1-\cos\theta)(1-\cos\phi)}{\sin\theta\sin\phi}$$
$$=\tan\tfrac{1}{2}\theta\,.\,\tan\tfrac{1}{2}\phi.$$

(*Cf.* Spheric chain in which $\alpha=\beta=\gamma=\delta$.)

PRACTICAL APPLICATIONS.

The general conic chain may be arranged, in practice, in either of the two forms shown below, or a combination of them ; the bearings fix plane AOB. The drawings are diagrammatic and in perspective.



Form I. FIG. 12. Form II.

Special Cases. (1) *Inversion of Hooke's Joint.* Most conveniently made as a combination of I and II. Plane BOC is fixed horizontal.

If AOB has angular velocity ω, ϕ will fluctuate between $\pm\alpha$, as shown previously, and hence we have a means of turning rotary into oscillatory motion.

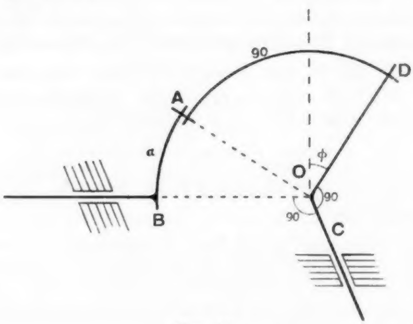
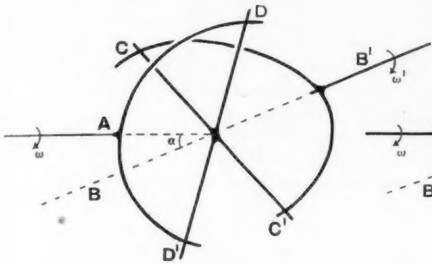


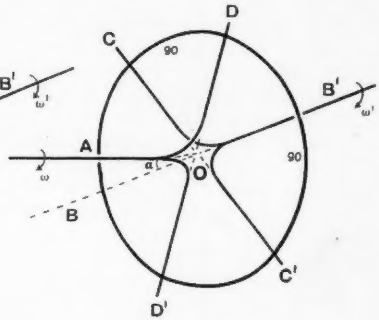
FIG. 13.

(2) *Hooke's joint.* For joining shafts not quite in alignment, but in the same plane. (Fig. 14.)

When arranged as in Form II, $CDC'D'$ may take the form of a split circular plate, in which case an Oldham's coupling can be fitted between the two halves of it, if the shafts to be joined are not quite coplanar.

In Form I, if there is sliding as well as turning at the hinges C, D, C', D' there may be a slight difference of plane between the two shafts to be joined.





Form I. FIG. 14. Form II.
(α is small.)

A model of Hooke's joint good enough for the drive to the propellor of a model boat, say, can be constructed from wire, preferably in Form II.

We have already found for Hooke's joint that

$$\tan\theta = \tan\phi \,.\, \cos\alpha, \text{ where } \theta = \omega t,\ \phi = \omega' t.$$

Differentiating, $$\frac{\omega'}{\omega} = \frac{\cos\alpha}{1 - \sin^2\alpha \,.\, \cos^2\theta}.$$

It may be convenient to have this velocity ratio in the form of an expansion ; this is most quickly obtained as follows :

$$\omega'/\omega = (1 - \sin^2\alpha)^{\frac{1}{2}}(1 - \sin^2\alpha\cos^2\theta)^{-1}$$
$$= (1 - \tfrac{1}{2}\sin^2\alpha - \tfrac{1}{8}\sin^4\alpha)(1 + \sin^2\alpha\cos^2\theta + \sin^4\alpha\cos^4\theta),$$

neglecting powers of α above the fourth, since α is small,

$$= 1 + \sin^2\alpha(-\tfrac{1}{2} + \cos^2\theta) + \sin^4\alpha(-\tfrac{1}{8} - \tfrac{1}{2}\cos^2\theta + \cos^4\theta)$$
$$= 1 + \sin^2\alpha(\tfrac{1}{2}\cos 2\theta) + \sin^4\alpha(\tfrac{1}{4}\cos 2\theta + \tfrac{1}{8}\cos 4\theta).$$

Thus

$$\omega'/\omega = 1 + \cos 2\omega t(\tfrac{1}{2}\sin^2\alpha + \tfrac{1}{4}\sin^4\alpha + \ldots) + \cos 4\omega t(\tfrac{1}{8}\sin^4\alpha + \ldots).$$

It might interest readers to consider for themselves the remarkable motions which occur as α approaches 90° in Hooke's joint and also the case in which $\alpha = 90°$, this being a particular case of those spheric chains in which $\alpha = \gamma = (180 - \beta) = (180 - \delta)$.

The Skew Isogram.

A mechanism for connecting two skew shafts, not coplanar, can be formed from the Skew Isogram ; the case for which the shafts are perpendicular is shown below.

Plane ABD is fixed. $\alpha = 90$, $\beta = \sin^{-1}(b/a)$.

The hinge-line at C is always perpendicular to the hinge-line at D.

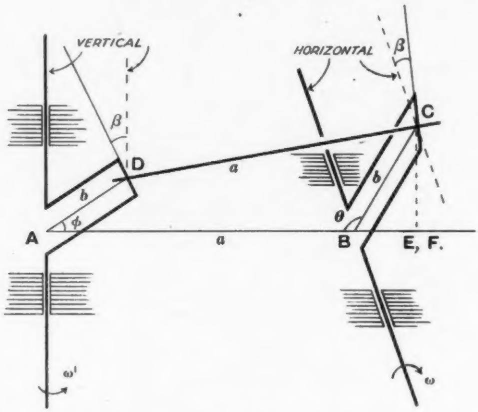


FIG. 15.

The mechanism is remarkable since it transmits a positive torque, even in the zero positions. (There are no " dead centres " and no " change points ".)

If ω' is constant (driver) and the fluctuation of ω is not to exceed $n\%$ of the mean ω, on account of excessive inertia forces,

$$\omega\sqrt{\left(\frac{a+b}{a-b}\right)}-\omega\sqrt{\left(\frac{a-b}{a+b}\right)}=\frac{n}{100}\cdot\omega,$$

and $a/b=\sqrt{\{4(100/n)^2+1\}}=200/n$, approximately.

If $n=25$, $a\approx 8b$, which is reasonable.
$n=10$, $a\approx 20b$, which is possible.
$n=5$, $a\approx 40b$, which is impracticable.

As also with Hooke's joint, a constant velocity ratio may be obtained by placing two similar Skew Isograms in series so that the second balances out the inequalities of the first.

A small working model can quickly be made from wire, using a cardboard box for the bearings.

Although the mechanism is not, perhaps, very suitable for power transmission due to the considerable stresses which are liable to be set up in the connecting rod and elsewhere, one would have thought it would have been more used as a movement in machinery, for there must surely be occasions on which it will provide a required motion in fewer pieces than any other mechanism. Perhaps, with the advent of plastics as materials for construction, engineering design may tend to become less rectangular and find room more easily for mechanisms such as this. R. H. M.

CAMBRIDGE MATHEMATICS IN WAR TIME.*

By J. G. Oldroyd.

Most of us who find ourselves still undergraduates in the third year of war have no great knowledge of Cambridge in peace time. Consequently, only the most superficial comparisons between pre-war and present-day conditions can be offered. It can be said, however, that while the private lives of both senior and junior members of the University have been very much affected by the war, mathematical activities here continue almost as normal.

In the first place, the same courses of lectures for Parts I and II of the Mathematical Tripos have been arranged as in pre-war years. There is no new short course for a war-time degree. When a man first comes into residence he can either read for Part I in one year and for Part II in his second and third years or, if suitably qualified, for Part II in his first two years in the hope of taking Part III in his third year. In choosing which course to follow, a student must now take into consideration his position as regards war service, since either programme may be interrupted by a man's being called up before he has completed his intended course. He will, where at all possible, prefer the more ambitious course if he can hope to be in residence for two years. On the other hand, if he is going to be called up at the end of his first year, he may naturally decide to attempt only Part I and postpone reading for Part II until after the war.

Although the lecture courses have not been altered, the lecture lists exhibit an obvious departure from the usual arrangements of pre-war years. For each lecture course, in peace time, there were generally two classes taken by different lecturers at the same times each week; for example, the lecture on *Differential Equations* would be given in the Michaelmas Term on, say, Tuesdays and Saturdays at ten o'clock by Mr. X in one college and by Mr. Y in another. Thus it was possible for each student to choose the lecturer whose line of approach (or rate of working) was more nearly in harmony with his own. But now that the number of lecturers in residence has been reduced by about half, only one don gives a particular series of lectures in any one year. In October 1939, when the number of students returning to Cambridge was quite a large proportion of the peace-time figure and when a remarkable number of freshmen made their appearance, this resulted in very large classes —for first-year students in particular. After two years of war the undergraduate population is much smaller and the attendance at any one lecture is not uncomfortably large. Fortunately the absence of lecturers on national service has not impaired the student's opportunities of having adequate personal supervision.

* [I am grateful to Mr. Oldroyd for this response to a suggestion of mine, that the *Gazette* would be glad of an article on the effect of war on the centre of English mathematics. Ed.].

Sometimes, of course, to arrange tutorials, it has been necessary to go further afield than one's own college if it has lost all or most of its mathematical dons. But the modern counterpart of the eighteenth century "coach" still plays an essential part in the life of a potential wrangler.

It is fortunately a rare occurrence that anyone reading for Part I or Part II is called up for national service during the academic year. There is therefore quite a stable community of undergraduates between October and June. On the other hand, a very large number go into the services or into war work at the end of each Easter Term. The number who are able to stay in residence for a further year to read for Part III after having, virtually at least, completed their degree course by taking Part II is very small indeed. Apart from one or two women students and a few men who are not eligible for military service there remain only those who will need an advanced mathematical training for the war work which they will be called upon to do. They may be summoned to begin this work at any time during the year. Consequently, not all of those who return to Cambridge after taking Part II will take the examination for Part III.

A Cambridge student does not specialise in any particular branch of mathematics until he begins reading for Part III of the Tripos. He then chooses to attend during the year perhaps seven or eight associated lecture courses on subjects which he intends to offer for examination. In war time the variety of lectures proposed by the Faculty Board has hardly been reduced. Occasionally, when a specialist has left Cambridge on war work, a blank has appeared in the lecture list, but generally another lecturer has been able to carry on a course usually given by one now on active service. Any technical work which a mathematician may be called upon to do is likely to demand a considerable knowledge of applied mathematics but, in the main, only such pure mathematics as is necessary as a background for the applied courses, that is, not more than is required for Part II. For this reason the advanced applied courses are much more popular with those preparing for the examination for Part III than the lectures of purely academic interest. Such subjects as the theory of statistics, elasticity and hydrodynamics attract attention as promising to be the most useful as a basis for technical work of national importance. Nevertheless, only one or two of the pure mathematical courses are entirely unattended (that is, are not given), although quite a number do not attract more than two or three students. The explanation of this paradox is that some of the Part II students attend these lectures as part of their general mathematical education without necessarily any intention of offering the subjects in an examination the following year.

In peace time a proportion of graduates would stay in Cambridge as research students after taking Part III of the Tripos. The probability of anyone doing this under present conditions is infinitesimal. After taking the examination one either ceases to deal

with any mathematics whatsoever or produces mathematics for some purpose useful to the war effort. In the latter case, no work is published unless it is likely to be equally useless to the enemy and to ourselves. Thus it is quite impossible to obtain a true picture of what Cambridge mathematicians are achieving either in the University or in their temporary appointments elsewhere. During the last few years before the war, there were about forty mathematical research students in residence—including a number from other universities in addition to those who had taken Part III in Cambridge. When war began they were in many cases allowed either to postpone submission of a dissertation for the Ph.D. degree until after the war or to offer one for examination at an earlier date than would normally have been permitted. The greater part of these students are now on active service or in important technical posts where research experience is essential. Only a negligible number of men are, during the war, beginning their careers as research students. Hence, although at the present time perhaps twenty men are registered as research students, not more than two or three are actually in residence. The senior members of the University who remain in Cambridge are doing the work of almost twice their number (not to mention their additional duties in private life) and are not able to produce the usual flow of original work. But it is safe to assume that enough new mathematics is appearing to keep alive every branch of the subject.

In many ways, mathematical thought and effort are being directed into the more applied and into special technical channels which are being explored in the interests of national security. It is not true that the war has annihilated the creative mathematician. Although his work has been diverted from its normal course, it has been stimulated in certain directions which are decidedly non-academic and which might be described as artificial; what precisely these new developments are will not be known, at least for the duration of the war. There is no reason to believe that the development of analysis and geometry—with the rather special significance which a Cambridge mathematician attaches to these terms—will not proceed with renewed momentum when Cambridge research students return to their peace-time habits. But whether the new war-like mathematical activities will leave their mark on the University as a whole or on particular mathematicians it is useless to conjecture. It is certainly true that many who are pure mathematicians at heart intend to return, from whatever technical work they may temporarily have undertaken, to swell the ranks of Cambridge geometers and analysts when the war is over.

No survey of the activities of Cambridge mathematicians would be complete without mention of the undergraduate societies and the Mathematicians' Tea Club. These are all functioning in spite of the relatively small undergraduate population. For the duration of the war, the four college societies are holding joint meetings in pairs. These meetings rarely give any indication of what new

mathematics is being produced as they usually take the form of a lecture by a don on some subject of slightly more general interest than the Tripos syllabus. The University mathematical society, *The Archimedeans* with its journal *Eureka* has gone only slightly further towards showing what is being done at the moment. However, some of the lecturers have produced refreshingly up-to-date ideas. Without a doubt *Eureka* would be more appreciated both inside and outside the University if it could go even further in this direction. In peace time the Tea Club was a flourishing society for Part III and research students. It is significant that the Club now has only three supporters who would be eligible for membership under normal conditions and has therefore been thrown open to junior undergraduates. Whereas, before the war, meetings consisted of lectures on modern work usually given by research students, the speakers to-day are almost without exception undergraduates or dons. Research students, representing as they do up-to-date mathematical ideas, are conspicuous by their absence. Thus, for the duration of the war, Cambridge Mathematics is maintaining in the national interest, a discreet silence. J. G. O.

1386. That night Mrs. Tom Jenkins came up to give me a polish in sums, written and mental. My father and mother, Ivor and Bron, and Davy were all round the table listening, and everybody quiet, pretending not to look.

We were doing very well, up to the kind of sum when a bath is filling at the rate of so many gallons and two holes are letting the water out, and please to say how long it will take to fill the bath, when my mother put down the socks she was darning and clicked her tongue in impatience.

" What is the matter? " my father asked her.

" That old National School," my mother said. " There is silly the sums are with them. Filling up an old bath with holes in it, indeed. Who would be such a fool? "

" A sum it is, girl," my father said, to soothe, quietly, " a sum, it is. The water pours in and takes so long. It pours out and takes so long. How long to fill? That is all."

" But who would pour water in an old bath with holes? " my mother said. " Who would think to do it, but a lunatic? "

" Well, devil fly off," my father said, and put down his book to look at the ceiling. " It is to see if the boy can calculate, girl. Figures, nothing else. How many gallons and how long."

" In a bath full of holes," Mama said, and rolled the sock in a ball and threw it in the basket, and it fell out, and she threw it back in twice as hard. " If he went to school in trews full of holes, we should hear about it. But an old bath can be as full of holes as a sieve and nobody taking notice."

" Look you," my father said to Mrs. Jenkins, " no more baths. Have you got something else? "

" Decimals, Mr. Morgan," said Mrs. Tom, " but he is strong in those."

" Decimals," said my father, " and peace in my house, for the love of God."

" Hisht," Mama said.—R. Llewellyn, *How Green was my Valley*. [Per Mr. V. I. Todhunter.]

CELESTIAL DISTANCES.*

By R. R. S. Cox.

When asked to give this lecture I was given to understand that this Society would like a change from the discussion of pedagogical methods and school problems, and could do with something that had nothing to do with teaching at all. In other words, I was to talk about stars and not to mention schools. Well, I promise to do that, though I must confess that the fact that members of the Society are teachers was at the back of my mind when choosing the subject. The schoolboy and schoolgirl and the man in the street know that celestial distances are very large ; in fact the adjective " astronomical " has been sadly degraded by being applied to such mundane affairs as the figures of the national debt—and a schoolboy of enquiring mind may ask how the astronomers measure such distances. Such an enquiry is often tinged with a certain scepticism —sometimes politely veiled, sometimes not. When you tell a member of the public that a certain nebula is many million light years away and that a light year is 6 billion miles, he is apt to adopt the attitude of mind indicated by the phrase, "You're telling me ", and to wonder what meaning such a monstrous distance can have : and how does the astronomer know, anyway?

So I thought that a few remarks about the methods by which such distances are estimated, and the way in which they confirm each other would be of interest to this Society as mathematicians, teachers and members of the public.

Before starting to describe the methods, however, I should like to say a few words about the value to astronomy of the results obtained. They have, of course, more than a mere curiosity value (useful though that may be to popular lecturers who like to create a cheap sensation). There is more in it also than the knowledge that so many metre rules placed end to end would reach from here to the sun or the Andromeda Nebula. That fact may be, and is, I believe, true : but it is not the fact about the metre rules which is significant. What is significant is that an object looks different at different distances—it looks smaller, fainter and appears to move more slowly the further it is away. If we are interested in the object itself the effect of the distance must be removed from our observations. A knowledge of the distance is highly important, therefore, if we are to separate appearance from reality. For example, if we did not know the distance of the sun we couldn't find its diameter : without that we could not find the value of gravity at its surface or in the case of a star the energy output of the star or the actual dimensions of a sunspot. Again, knowledge of the large distances of the globular clusters or the remote nebulae is essential if one is to know their size, or to compare their sizes with each other, or to study their arrangement in space. So we may conclude that the investigation

* A lecture to the Sheffield Branch of the Mathematical Association, 10th June, 1941.

of these distances is an undertaking of considerable value from a scientific point of view.

The fundamental method, of course, is in principle the same as that described in textbooks of trigonometry for finding the distance

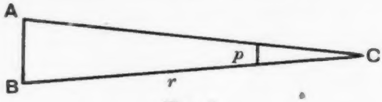


FIG. 1.

of an inaccessible object. We view the object C from two stations AB. From the angles and the length of the base line, the distance can be found. In astronomy, however, the triangles are always very long and thin, and the practical difficulty comes down to that of measuring with sufficient accuracy the small angle ACB. This angle is called the parallax, and is usually measured in seconds of arc.

Within the solar system, the earth's equatorial radius is taken as the standard base or unit of distance. Outside the solar system,

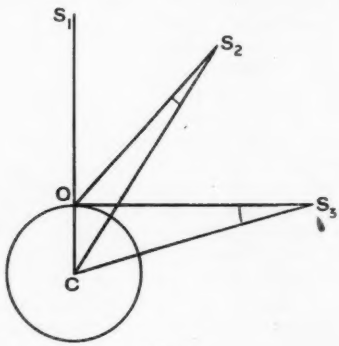


FIG. 2.

the mean radius of the earth's orbit is taken as unit. Within the solar system, for example, the position of a body is usually predicted as viewed from the earth's centre. When it is overhead, as at S_1, apparent and predicted directions coincide. When at S_2, there is a difference $= \angle OS_2C$. When the body is on the horizon, the angle is a maximum and is then called the horizontal parallax, and is the angle subtended at the body by the earth's radius. The effect is to make the body appear lower in the sky than if the effect were neglected. In the case of the moon the angle is considerable—about 1°, which is double the moon's diameter. The effect is so great that to ignore it introduces large irregularities in the moon's apparent motion. One of the important pieces of work which Ptolemy did

was to elucidate this effect ; and by comparing the expected with the observed position of the moon to calculate its parallax with reasonable accuracy.

The Greeks therefore knew the distance of the moon ; but to find that of the sun was beyond them : in fact the angles cannot be

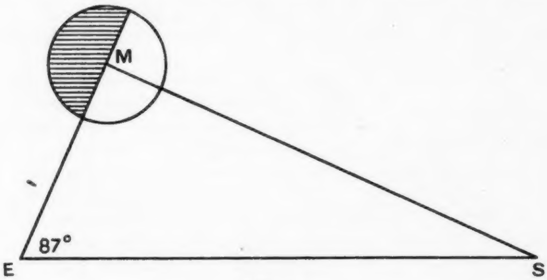


FIG. 3.

measured without a telescope. A brave attempt, however, was made ; and the principle may be of interest. *Aristarchus* (281 B.C.) found the ratio of the sun's to the moon's distance by observing the angular distance between the sun and the moon at the quarter. When at exactly half-moon, $\angle EMS = 90°$. Aristarchus observed, as he thought, $\angle MES = 87°$. He also proved that cos 87° lay between 1/18 and 1/20 and deduced that the sun was between 18 and 20 times as far away as the moon. This would make the solar parallax about $60'/20 = 3'$, making the sun $240{,}000 \times 20$ or less than 5,000,000 miles away. This value was accepted by Ptolemy and copied by all astronomers without question for over fifteen centuries. The unquestioning acceptance of a theory like the Ptolemaic Theory which fixed the earth at the centre for 1600 years is to our critical minds bad enough, though understandable ; but the acceptance of a definite numerical figure without any attempt even to improve its accuracy seems to me far worse. The principles used by the Greeks were perfectly correct : only the measure of 87° was hopelessly wrong. But it was not until 1650 that a certain Wendelin thought of repeating the observation : he got $89° \, 45' = \cos^{-1}(1/229)$, leading to a solar parallax of 14″ or 58 million miles. Note that this method is really the parallax method using the radius of the moon's orbit as base line. But, of course, the moment when the moon is halved cannot possibly be judged with sufficient accuracy, and the method is in practice useless.

The first attempt to obtain the solar distance by a method capable of an accurate result was made in 1672 by Cassini and Richer : Richer made an expedition to Cayenne in S. America ; Cassini stayed in Paris : and both observed Mars over a period of several months. It is said that when Richer returned, after having had

various exciting adventures in the jungle, and using a log hut roofed with palm leaves as an observatory, he was hailed as a hero, much to the annoyance of Cassini, who, though he stayed at home, felt himself to be the real leader of the enterprise. They used a method which has been much employed since, namely the determination of positions relative to a comparison star, in this case ψ' Aquarii, the light from which is effectively parallel. The angular distances between star and planet were measured at the two stations, and the parallax of the planet deduced.

Given the distance of Mars, all other distances in the solar system follow, the proportions between the various distances being known, through the work of Kepler. The planet Mars was used (*a*) because it is nearer than the sun and therefore has larger parallax ; (*b*) the position of the limb can be more accurately observed with a planet than with the sun. Cassini and Richer found the angle subtended at Mars by the line joining their two stations to be 15″ and deduced that the solar parallax $=9{\cdot}5''$ or 86 million miles.

An improvement in the method, introduced later, has the advantage that it avoids long journeys. Instead of taking ship to Cayenne, you just wait a few hours until the earth's rotation has carried you a few thousand miles away and then take another observation, with the further advantage that you are using the same instrument.

A further improvement was the use of one of the minor planets instead of Mars. The minor planet favoured for this purpose is Eros, which can come within 14,000,000 miles of the Earth, and therefore is better than Mars, which at its nearest is 34·6 million miles away. It appears also almost as a point of light in the telescope. Observations were made extensively in 1901 from 58 observatories. The reduction of these observations is a big undertaking, and the result was not published till 1910 (8·806 or 7″). Eros came round again in 1931, in a more favourable position than in 1901 ; and very extensive preparations were made to observe it. The final result has not yet been published ; in fact, there have been some difficulties. Eros is not actually a point ; it is 15 miles in diameter. Its brightness varies periodically in a period of 5 hours 16 minutes ; with large magnification it appears to have a shape like a figure eight, and is evidently a body of irregular shape, or irregularly marked, rotating in a period of 5 hours 16 minutes. It has even been suggested that it consists of two or three bodies revolving about each other in complicated orbits, and that the centroid of mass differs from the centroid of brightness in such a way as to make it unsuitable for parallax observations. But this view does not seem to be confirmed. The previously accepted value of the sun's parallax was 8·80″, but the Eros observations seem likely to yield 8·795″.* Now 8·80″ gives 92·87 million miles as the sun's distance while 8·795″ gives 92·94 million miles, so the two observations give values of the sun's distance differing by 70,000

* The definite value, subsequently published, is $8{\cdot}790'' \pm 0''{\cdot}001$.

miles. I don't think that anyone now has any doubt that the figure $8{\cdot}80''$ is correct to two decimal places : but it is very satisfactory to have confirmation from quite a different source. This source is a group of methods which may be called velocity of light methods.

We are familiar with the observations of Römer, who in 1675 noted that eclipses of Jupiter's satellites took place at times differing from those calculated, the later the further Jupiter was away, and explained this by the greater time the light took. He reckoned that it would take 22 minutes to cover the diameter of the earth's orbit ; and assuming solar parallax deduced a value for the velocity of light. The method can be reversed, and with a velocity of light determined by terrestrial experiments used to find the solar parallax. The method is not capable of any accuracy, since the satellites become gradually immersed in the shadow, but it affords an indication of other possibilities than the trigonometrical. The first that was used consisted in measuring the aberration of light received from the stars, and so determining the speed of the earth in its orbit. A better method is by use of the Doppler effect. The component of velocity of a star in the line of sight—the radial velocity as it is called—can be measured by the shift of the spectral lines to the blue or red. This shift is given by the equation $v/c = \delta\lambda/\lambda$, where λ is the wave-length and v the velocity component of the star. In the practical reduction of the observations account has to be taken of the motion of the earth. The radial velocity measured is relative to the observer and is therefore affected by the rotation of the earth and by its orbital motion. The rotation effect v_r is very small ; at its maximum it is about 0·4 km. per sec. The orbital velocity effect v_o can reach 30 km. per sec. Now the observed radial velocity $v_m = V + v_o + v_r$, where V is the velocity of the star relative to the sun. The value of v_r can easily be calculated ; v_o varies with the season. Plates are taken at different times of the year, and so we obtain V and v_o. Recently at Mount Wilson they were trying out a new spectrograph with the 100 in. telescope, embodying a new aluminium on pyrex grating, with great concentration of red in the first order and of ultra-violet in the second order spectrum, and a Schmidt camera. The star Arcturus is often used for testing spectrographic apparatus because its spectrum has sharp and narrow lines ; they used it to test this instrument ; and had 37 plates taken at different times of the year. So they thought they would use them for the solar parallax. After reducing them they found $p = 8{\cdot}805'' \pm 0{\cdot}007$ and the velocity of the star as 5·621 km. per sec. $\pm 0{\cdot}005$. Note that this is almost a casual by-product of the testing of the instrument, but it gave a reasonable result with a fraction of the labour used in connection with the Eros observations. The percentage error is regarded as large and the sources of error probably present include the character of the arc producing the comparison spectrum and the temperature control of the instrument. If these can be removed and more plates taken, a very good value of the parallax may be obtained. Note that it is not a case of

measuring a small deviation from another effect: v_0 is in this case up to 26 km. per sec., while the radial velocity is only 5·6.

I hope I have now said enough to convince you that the sun is about 93 million miles away. Let us next pass on to the distances of the stars. Here we meet a more difficult problem in that the angles to be measured are even smaller. We use the radius of the earth's orbit as base line; the parallax of the nearest star is only ·75", so that the angle to be measured is $1\frac{1}{2}$". Many attempts were made to detect this effect.

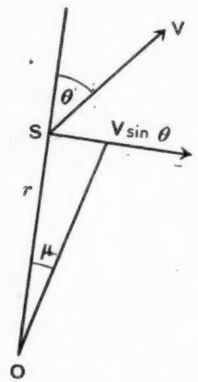


FIG. 4.

Tycho could not find it, and deduced that the earth did not revolve about the sun. Others thought that they had found it, but their results were due to experimental error: the more successful they thought they were, the more inaccurate were their instruments. It is important to note one difficulty. The obvious method was to tackle the easiest problem first, that is, the nearest stars. But how do you know the best star to choose? Hundreds of thousands are visible even in a moderate telescope. Which are the nearer ones? The difficulty of choice held up for a long time the solution of the problem. Bradley (1800) reckoned that his instrument could detect a parallax of $p = 0{\cdot}5''$, but no one could find such a star. Actually there are only three known stars with a parallax greater than 0·5". They naturally used brightness as a criterion for nearness; actually it is a very bad one, because the stars differ enormously in intrinsic brightness. For example, Rigel is very bright indeed, but it is too far away to measure. Of the 18 nearest stars, 9 are invisible to the naked eye. A better criterion is proper motion, or angular shift across the line of sight. In early times the stars were supposed to be fixed; the shape of the constellations has not altered in historic times, as far as the naked eye can perceive. But actually slow angular displacements can be measured, by taking plates, at, say

ten-year intervals and measuring the shift. The angular displacement in a year is called the *proper motion*, μ. In Fig. 4 we have

$$\text{radial velocity} \quad = V\cos\theta = R,$$

$$\text{tangential velocity} = V\sin\theta\,;$$

and hence $$\mu = V\sin\theta/r,$$

and the parallax $$p = (4{\cdot}74\mu/V\sin\theta)''.$$

Now values of V are much more alike than values of the intrinsic brightness. Hence a large μ is a good criterion of a large p. The man who first realised this was Bessel, who selected 61 Cygni, the star with the largest proper motion then known (5·2″), and found a parallax of 0·314″ (modern value 0·285″) in 1838. In the same year Struve in Russia and Henderson at the Cape also succeeded with other stars, and the great problem was solved. Bessel and Struve used the differential method, measuring the angle between comparison and parallax star. Since then the greatest progress has been made by the use of photography. A series of plates of a star is taken at different seasons and the distance between the star and a number of surrounding comparison stars, fainter and presumably more distant than the parallax star. One has to separate the proper motion from the parallax, which is easy; the proper motion goes on steadily, the parallax oscillates seasonally. Extreme accuracy is needed and a strict guard against systematic errors—for example, the magnitude error which occurs when, as is usual, the parallax star is much brighter than the comparison stars. If the guiding of the telescope is not perfect, the images are drawn out and the bright star may record a deviation and the faint star not. To meet this the bright star is dimmed by rotating a sector in the path of the light from the parallax star.

The trigonometrical method is fundamental; but unfortunately most stars are too distant for it. The average P.E. is about 0·007″. A star with parallax ·04″ can be measured with an accuracy of about 15%. At twice this distance one can get some idea of the parallax; but farther than that the method becomes unreliable. What is the scale of these distances? Here we introduce a unit, the *parsec*, which is the distance equivalent to a parallax of 1″. Hence a parallax p is equivalent to a distance of $1/p$ parsecs and

$$\begin{aligned} 1 \text{ parsec} &= 206{,}265 \text{ times the sun's distance} \\ &= 3\tfrac{1}{4} \text{ light years} = 19{\cdot}16 \times 10^{12} \text{ miles.} \end{aligned}$$

The nearest star has $p = {\cdot}076''$, and so is 1·3 parsecs distant. The greatest distance measurable is, say, 50 parsecs, and about 2,500 stars have been measured in this way. Can we get out any further?

The most spectacular results with which readers of popular books on astronomy are familiar are the distances of the Spiral Nebulae derived from the study of the Cepheid variables. I don't wish to go into this matter now, but rather to spend the time on some of the

humbler foundations, not mentioned in popular writings, on which the results rest.

The relative distance of Cepheid variables in star-clusters and nebulae can be derived from the results of observations of their apparent brightness and the period of their light variation. To pass from relative to absolute distances it is necessary to know the absolute distance of one, or better, of a group of Cepheids. Unfortunately no known Cepheids are near enough for their distances to be measured with sufficient accuracy trigonometrically, and other methods must be used.

I have time only to mention two examples of a class which may be called "proper motion methods", one of which is of value in connection with the Cepheid variables.

Moving clusters. Certain groups of stars or clusters have proper motions that are conspicuously similar. For example, in Ursa Major

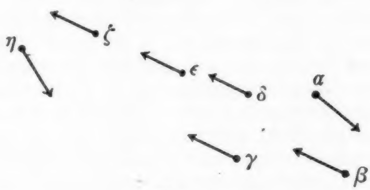


FIG. 5.

all except α and η are moving in the same direction with similar proper motions. The radial velocities are similar also; and it seems evident that we have a group of stars moving parallel to each other with common velocities and probably arising from a common origin. When such groups cover a large area one would expect, from a perspective effect, the apparent motions to converge; and this is actually observed in one or two cases. In these cases, the distance of each star in the cluster can be found with considerable accuracy.

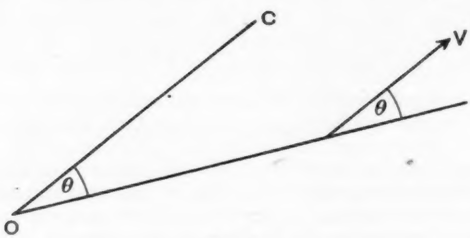


FIG. 6.

The stars are moving parallel to each other in a direction parallel to that from the observer O to the convergent point, C. The radial

velocity R is equal to $V \cos \theta$. Now R and θ can be measured, and V found. It is the same for all stars of the group. Then the transverse velocity T is equal to $V \sin \theta$ and the proper motion is

$$V \sin \theta \,.\, p/4{\cdot}74,$$

and therefore $p = 4{\cdot}74\mu/V \sin \theta$, and p is found for each individual star.

About 72 members of the Ursa Major cluster are now known. There are a few other clusters of this type : Perseus, Scorpio, Centaurus. For these the parallaxes agree well with those found by the trigonometrical method, and this forms a valuable confirmation of both methods. Notice that in this method we are using the motions of the stars themselves as base line.

Another, and most important, method uses the motion of the sun relative to the neighbouring stars. It was discovered by Herschel that the motions of the stars tend to diverge from a certain point in the sky and then converge to the opposite point. This effect is evidently due to the Sun's motion relative to the average of the stars observed ; the speed is 19 km. per sec. (4 astronomical units per year) towards a point near Vega. If only the stars would stay still we could use this to give a long base line. What we can do is to average out their individual motions and obtain a mean parallax for a group of stars. It is this method which was used to determine the mean distance and the absolute magnitude of certain Cepheids which fixed the zero point of the period luminosity curve of Cepheids. It forms an essential step in the chain of processes leading to the distances of the remote nebulae.

One final remark. These distances have been obtained through one or two fortunate circumstances. If we had a region about us where stars are sparsely scattered so that the nearest were twenty times as far away as they actually are, then trigonometrical parallaxes would not have been found so early, perhaps not even in 1941. If, again, the Cepheids had not happened to be so intrinsically bright, they would not have been visible at all at such great distances. The astronomers have truly been clever ; but they have also been aided by fortune. R. R. S. C.

1387. The sudden change in direction against the pull of gravity may tear the wings off a machine. Pilots call this giant force developing for a few seconds in the pull-out " G " (for gravity), and according to the speed, weight and other factors in the falling machine, this force can reach 9 G the limit the human frame can bear. In simple terms " 9 G " means that for a few moments the pilot weighs nine times his own weight. A ten stone man weighs more than half a ton!—*John Bull*, February 3, 1940. [Per Mr. W. Martyn.]

1388. Chinese authors are apparently indifferent to arithmetic and figures in general. After comparing the best available editions of Yüan's works, I still cannot find the reputed " twenty-three " conditions. It really doesn't matter whether one's figures are correct. Mathematical exactitude worries only a petty soul.—Liu Yutang, *The Importance of Living*, p. 322. [Per Mr. D. J. Finney.]

VITAL MATHEMATICS.*

BY R. S. WILLIAMSON.

MATHEMATICS existed prior to the mathematician. Before Man calculated, Nature multiplied and divided; her products were based on geometrical forms. Mathematics, in reality, is not a creation of the mathematician. It is a vital spirit, immanent in Nature and Man, and expressed in the forms and processes of Nature and the manifold works of Man. This spirit functions only partly through the mathematician, who limits both its sphere of operation and its method of working. These limitations constitute a serious difficulty for the pupil by cramping the expression of his personality. His love of beauty is ignored, his social sympathy is untouched, his sense of humour lies dormant, his interest in Nature is not aroused, his imagination has little outlet, his artistic, musical and religious tendencies find no contact. On the whole his emotions are starved, his intellectual development is limited and his spiritual life is uninfluenced. These are hardly the conditions for the successful teaching of any subject. Nor does such limited mathematics contribute adequately to the general culture of the pupil. Its limitations are also a serious obstacle to the integration of the school curriculum, one of the most pressing problems.

School mathematics should not consist merely in studying the mathematician's product and learning to use the intellectual machine he has invented. It should be the study of the vital spirit of mathematics, in whatever forms it is expressed, in all spheres of knowledge and experience which contribute to the pupil's education.

This means that the whole school curriculum would be involved. Use would be made of the history of mathematics, not just to provide interesting snippets, but to illustrate the nature and development of civilisation. The pupil would also learn that mathematics is a common activity throughout the world and that international co-operation is to some extent based upon it. Thus a sympathetic insight into and a sense of unity with world civilisation, both historical and geographical, would be cultivated. Artist, writer and musician may express mathematical ideas in their own ways. From their productions the pupil would find that mathematics is not limited to the material affairs of life, but may serve beauty and truth in the more cultural aspects of human achievement. He would realise, too, something of the wholeness of civilisation's intellectual tradition. In Nature he would find the spirit of mathematics in alliance with those of beauty, utility and economy, and would learn that Man builds up a world of his own on the principles of Nature. This should lead to the appreciation of Nature and help to restore her to her proper place as a healing and inspiring element in civilisation. Religion comes out of the nature of Man and has always been a feature of his civilisation. This may be brought home

* This article refers particularly, but not only, to Senior School Mathematics.

to the pupil, for the spirit of mathematics has played a part in the development and expression of religion through, *e.g.* the number mysticism of Pythagoras, the philosophy of Plato and the application of geometry to religious architecture. The pupil may thus realise that civilisation will only serve the full needs of Man when it embodies spiritual ideals. The general aim of all this work would be not only to broaden the basis of mathematical education and thus help to develop the whole of the pupil's capacities, but also to bring out fundamental principles of civilisation. These principles might form a basis for unifying the school curriculum. Conventional mathematics, viewed historically and philosophically, would illustrate some of them.

It must be emphasised that something more than the process of correlation, as usually understood, is being suggested. In correlation, when two masters deal with the same subject-matter, each presents it from a different point of view; the reconciliation or unification is left to the pupil. It is suggested that the two points of view be shown to be different aspects of the same wider viewpoint. A simple illustration should make this clear. There is a poem describing birds on a tree; so many are there, a number fly away, some remain. Replace one of these numbers by a note of interrogation and we get a simple arithmetical problem. The verse has served its purpose so far as conventional mathematics is concerned. Mathematical treatment should go further, enquiring why the poet chose the particular numbers. This would bring out that he was under the discipline of arithmetic as well as of the rules of rhyme and rhythm, and, what is most important, that his arithmetical attitude of mind is an organic part of his whole outlook. This latter point would be reinforced by showing his calculations as an element in his whole poetic attitude of endeavouring to describe a scene accurately; so poet and arithmetician are one. A common outlook is thus established for the English master and the mathematics master which is wider than the viewpoint of either. It embodies the idea, fundamental to Man and civilisation, that the individual draws on his resources *as a whole* in expressing himself and making his contribution to culture. It is based on that unity of knowledge and that unity of personality which the specialist system of teaching tends so much to undermine.

The same common outlook may be seen reflected in the association of mathematics with music, religion and art. In a simple folk-tune Man embodies the idea of multiplication in the repetition of a musical phrase, and that of fractions in the varied use of crochet, quaver and semi-quaver to express beauty of form and melody. For religious worship he builds a church on a rectangular plan, and uses the circle in various ways to give it beauty and atmosphere. The converging lines of its tower and spires uplift both his outer and inner vision to the Infinite. In an art design he may also express beauty of form by arranging geometrical figures according to artistic principles, as in a crochet or needlework pattern or in the tracery of windows.

These instances provide similar material to that of the poem, and the teacher may deal with them on similar lines.

With regard to conventional mathematics this would be presented to the pupil not merely as a means of solving arithmetical problems and doing geometrical constructions, but as an activity, a partial mode of life, which Man, following Nature's inspiration, has developed for himself in building up civilisation. Into this activity, this mode of thought, feeling and action, the pupil is to be initiated. He would solve problems and do constructions as before. He would do so, however, not only for his own immediate ends, but as part of the experience which comes to him in studying and learning to play Man's part in civilisation. From the significance thus attached to this work he might be brought to see important principles of civilisation reflected in it. Consider, for instance, the idea of integration expressed in "Little strokes fell great oaks", and exemplified in the building of the Egyptian pyramids, a bird's nest or a modern battleship, as well as in the painting of a picture or the production of a piece of needlework. Here is a principle which Man, individually and collectively, has employed freely in the development of civilisation. Arithmetic abounds with illustrations of its use, *e.g.* to find 257×63 he performs several simple multiplications and adds the products. The pupil should be led to realise in this procedure not only an arithmetical rule, but a universal principle which regulates and inspires the separate and corporate life of Man. Again, consider the idea of Man the toolmaker. This idea, in a limited form, is generally presented in the history lessons, but it is implicit in all subjects of the curriculum, and has an exceedingly wide implication in relation to modern civilisation. In mathematics it is illustrated by the instruments of geometry and, in a wider sense, by numerous devices of arithmetical calculation, *e.g.* the formula, as well as by the progressive development of the whole subject. The mathematics master may therefore play an important part in inculcating the whole idea, and in throwing light on one of the most pressing problems of modern civilisation, the relation between Man and the "machine". In this philosophic treatment the historical method would be invaluable. For example, in the first of the two instances just given the integrative principle could be shown more clearly in connection with some historic method of multiplication, *e.g.* that practised in the Middle Ages, than with our present method. Similarly, in the second instance the use of the abacus in early times would illustrate excellently the "tool" idea. In both these cases something more than the history of mathematics is intended. The mathematics should be shown in a historical setting, that is with a background of contemporary circumstances. This background might also be geographical. But sometimes a geographical background alone would be needed, *e.g.* in an account of the use of the soroban, the Japanese abacus. Whatever the setting, Man, particular or general, should occupy the foreground, applying some universal principle in connection with mathematics. Thus a com-

mon outlook would be established for mathematics, history and geography as in the case of mathematics, literature, music, art and religion.

I need not dwell on the case of Science, the study of Nature, which, particularly in its biological aspect, is very suitable for association with mathematics reorientated as suggested. It must suffice here to carry a little further the idea, mentioned in par. 3, that Man has built up his civilisation on principles found in Nature. This may be well illustrated through geometry. The pupil may study geometrical forms found in Nature and learn how Man replaces her imperfect shapes by regular figures, the crude circle of the flower by a uniform curve, a stalk by a straight line. He may consider flowers or leaves from a particular kind of plant, and so observe Nature's varied efforts to produce perfect shape. From this he may be helped to realise something of Man's own struggle to " attain the highest ", and to see in the perfection of regular geometrical forms a symbol of the ideal Man pursues.

The reader must beware of thinking that mathematical material would be drawn only from the present subjects of the school curriculum. Paragraph 2 indicates otherwise. Thus, for example, though aesthetics and sociology may not be suitable subjects for school instruction, they may quite well furnish, for the broader treatment of mathematics, viewpoints which would be valuable in the education of the pupil. For instance, Man's search for beauty, and the part of tradition in human society are two topics, at present neglected in school, which would serve this purpose. The criterion must be the aims of mathematical teaching, which have been previously indicated. These may be set down formally thus :

1. To help in the development of *all* the pupil's innate capacities, and of his personality *as a whole*. To contribute adequately to his general culture.

2. To inculcate principles which Man has applied in building up civilisation and reveal them as fundamental principles in the development of mathematics. To help to show them as unifying elements in the school curriculum.

It may be argued that these suggestions involve placing an undue burden on the mathematical department of the school. The increase of subject-matter, however, need entail little difficulty for the pupil, for, from the writer's experience, he would work with increased interest and show much greater working capacity. The new work, which need not be great, would also provide an interesting basis for some of the conventional mathematics as well as illuminate it.

It is clear that the sum, or problem, at any rate in its present form, is an unsuitable medium for this work. But something as universally useful as a means of teaching is necessary. It is suggested that reading matter should be used. Thus a " literature " of the subject would be needed. The stories below are an attempt to indicate something of the nature of this " literature ". As for

the teacher, it would seem natural to ask him to adjust his mind to the rest of the curriculum. His academic course, whatever its value in developing special aptitudes, tends to stultify general development. He must begin to recover from this, preferably during his professional training, certainly in the school, for his own sake as well as the pupil's. In developing a philosophy of life for himself he must, if thorough, work out his own attitude to art, literature, music, history, science and religion, as well as mathematics, in the light of universal principles. In doing so he will be adopting the standpoint suggested in this article. He should find that it invests his subject with a new interest and brings a humanising influence into his work.

There is no doubt that school mathematics, reorientated in the way suggested, would spring to life from the ashes of its dead self. It would stimulate the life of the pupil through a full range of human capacities. It would illumine the life of Man in time, space and eternity. It would be revealed as a living spirit. Truly School Mathematics would then be what it ought to be: VITAL MATHEMATICS.

Vital mathematics illustrated.

In the following stories I have tried to show that the idea of presenting fundamental principles of civilisation to young pupils in connection with the teaching of mathematics is not merely a theoretical aspiration, but a practical possibility. The stories together touch most subjects of the school curriculum, and cover practically all the inadequacies of mathematics mentioned in the first paragraph of the previous article. They both require a *thorough* knowledge of the relevant conventional mathematical detail to appreciate them fully.

Kam and his dog.

It happened more than five thousand years ago, but long after Man had first learnt to count. One fine day Kam arranged to work in his fields with some friends. There were five of them altogether, Kam and four friends. Kam determined to have no difficulty about sharing food this time. So he counted out carefully seven cakes for each worker, including himself.

It was harvest time in the flat lands of the Nile valley. The workers had to gather in the rich brown crop of waving corn. Just before they began Hesh turned up. "Oh! I'll give you a hand", said he, with his usual generosity. And soon they were all working with a will in a large field of barley.

By and by they sat down for their meal. Kam produced the cakes and they began sharing them out. Each took one in turn in the usual way. Kam had quite forgotten that he had arranged to share them out exactly. But he quickly remembered. For soon only five cakes were left for six workers. So the sharing came to a stop.

Hesh said he would not have another cake. But Kam reminded him of their fishing trip. Did not the gods then show them how to overcome a difficulty of sharing? So he took his knife and cut each of the cakes into " sides ", or halves. Each person took one half. But to Kam's surprise two cakes, ⦶, ⦶, still remained. So again the sharing stopped.

At first Kam was completely puzzled. He put on his thinking cap. Soon he uttered a cry of delight. The gods were not wrong. Quickly taking his knife he cut each cake again along a diameter. This time the cut was from left to right of the cake, thus ⊕. He had halved each half and found a quarter.

Once again each worker took a piece. But again Kam was disappointed. Two quarters, ◗, remained. You can imagine how confused he was. Fortunately just at that moment a " Yap Yap! " was heard. It was Kam's dog. He was hungry and was getting fractious.

" Hullo ", said kind-hearted Hesh. " Nekht is counting the pieces." Kam took the hint and threw the rest of the cake to Nekht. And everybody had a good laugh. For Nekht had solved the problem.

Note. The above story is the third of three stories intended to throw light on one of the most fundamental problems of Man and his civilisation, viz. an opposition which appears in various forms, *e.g.* between heart and mind, emotion and reason, humanism and science, idealism and materialism, Man and machinery. The story may be taken at its face value as showing how Kam's machinery needed tempering with the human feeling of Hesh. Nekht, uttering his inner feelings in arithmetical form, provides the solution. The story is also an allegory, a plea for the humanistic teaching of mathematics.

It will be noticed that there runs through the story the theme of justice, a fundamental principle of democracy, in the form of equitable distribution of food in a small community. A third principle of civilisation is the belief in a god or gods. The setting of the story is both historical and geographical.

And this shall be the sign.

The circle is a lovely shape. It has no sharp corners or broken lines like a square or oblong. A single line makes the figure, with no beginning and no end. This line is not straight and monotonous. Nor does it make sudden darts inwards or outwards. It curves steadily round, always with the same graceful curve. Like the Magi of old it has a guiding point. But it remains always the same distance away. The circle has no need to seek perfection. Is it not itself perfect?

Early Man thought so. He looked up by day. There it was in the heavens, shining beneficently—the Giver of warmth and light. Without these, animals, trees and plants would die. Man himself would perish. Was it not the Source of Life? Surely that shining

the teacher, it would seem natural to ask him to adjust his mind to the rest of the curriculum. His academic course, whatever its value in developing special aptitudes, tends to stultify general development. He must begin to recover from this, preferably during his professional training, certainly in the school, for his own sake as well as the pupil's. In developing a philosophy of life for himself he must, if thorough, work out his own attitude to art, literature, music, history, science and religion, as well as mathematics, in the light of universal principles. In doing so he will be adopting the standpoint suggested in this article. He should find that it invests his subject with a new interest and brings a humanising influence into his work.

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circle was God Himself, God the All-powerful, the All-knowing, the All-wise.

But later, wise men of ancient nations said, " No, the sun is not God. It is God's handiwork, and God's instrument. Through it we may know God ".

And other handiwork of God they saw. By day, there were beautiful flowers with circling petals, and the rounded boles of forest trees. By night, they saw the circle moon that waxed and waned, and the rounded bowl of star-lit heaven.

Then they said, " These things are good and beautiful. It is the goodness and beauty of God, God the Great Creator, God who always was and always shall be. So the circle, the line without beginning and without end, shall be the sign or symbol of God."

And Christian men to-day use the sign of those ancient nations. In their pictures Christ is shown in the form of a man. But round his head is a halo, a circle of light. For Christ, they say, is not only Man but God Himself.

NOTE. My primary aim in this story is to present religion as an element in civilisation. The matter is treated historically, but the treatment may appear vague, so perhaps I should say that in other stories, intended to precede this one, I have illuminated terms such as " early man " and " ancient nations ". The meaning of " sign or symbol " may be brought out in connection with notation, and of " handiwork " and " instrument " from the geometry. Geometrical ideas, technical and everyday, are introduced freely, partly for the sake of revision. The story touches Scripture (Magi) and Nature Study (of earth and sky), and it should call into play the artistic or aesthetic side of a pupil's nature. R. S. W.

1389. In the space of one hundred and seventy-six years the Lower Mississippi has shortened itself two hundred and forty-two miles. That is an average of a trifle over one mile and a third per year. Therefore, any calm person, who is not blind or idiotic, can see that in the Old Oolitic Silurian Period, just a million years ago next November, the Lower Mississippi River was upwards of one million three hundred thousand miles long, and stuck out over the Gulf of Mexico like a fishing-rod. And by the same token any person can see that seven hundred and forty-two years from now the Lower Mississippi will be only a mile and three-quarters long, and Cairo and New Orleans will have joined their streets together, and be plodding comfortably along under a single mayor and a mutual board of aldermen. There is something fascinating about science. One gets such wholesale returns of conjecture out of such a trifling investment of fact.—Mark Twain, *Life on the Mississippi.* [Per Mr. F. J. Wood.]

1390. " A circle with a radius of twenty miles includes an area of a hundred and twenty-five square miles. Rather an extensive area to search. . . . But, in the course of his journeys he covered the same route four times. If my arithmetic is correct, twenty divided by four is five. That means that the base can't be more than five miles from here. A circle of five miles radius covers an area of thirty-one square miles. A considerable reduction in the size of the haystack in which you've got to look for your needle."—Miles Burton, *The Milk-churn Murder*, pp. 49, 53. [Per Mr. D. J. Finney.]

THE REGULAR OCTOHEDRON.

By W. Hope-Jones

Of the five regular solids, the octohedron is conspicuous (though perhaps not more so than the cube) for its connections with the others : one of these is new to me and possibly not as well known to others as it deserves to be.

Cube. The commonest, and probably the best, way of presenting the octohedron (" regular " will be taken for granted in much that follows) to the beginner is by joining the middles of the faces of a cube. This relation (like that between the icosahedron and dodecahedron) is reciprocal : that, is, a cube can also be formed by joining the middles of the faces of an octohedron.

Tetrahedron. The fact that the tetrahedron and octohedron have supplementary angles (of $70° 32'$ and $109° 28'$) between adjacent faces is well known to adult mathematicians, but generally unknown to the young, chiefly, I think, because it is not widely enough realised how easy is the proof of it, well within the range of the average boy of 13.

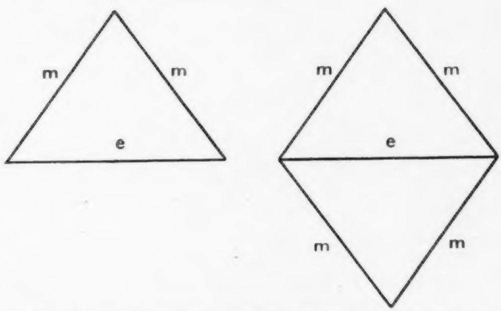


Fig. 1.—Plane sections of tetrahedron and octohedron.

Take a tetrahedron and an octohedron having equal edges (e), and therefore equal medians (m), of their twelve triangular faces. Paint round each a central plane section containing the angle under consideration : on the tetrahedron the section will be an isosceles triangle of sides m, e and m ; on the octohedron a rhombus whose sides are m and short diagonal e. Half of this rhombus is congruent to the triangle : therefore the angle between adjacent faces of the tetrahedron = the acute angle of the rhombus, which is supplementary to the obtuse angle, which is the angle between adjacent faces of the octohedron.

From this it follows that sticking on to a face of an octohedron a tetrahedron of equal edges has the same effect as producing that face's three neighbouring faces to meet at a point outside the octo-

hedron. Treating all the octohedron's faces thus produces a very graceful figure called the "stellated" or "star-pointed" octohedron, which can be cut out on the flat in one piece. It can also

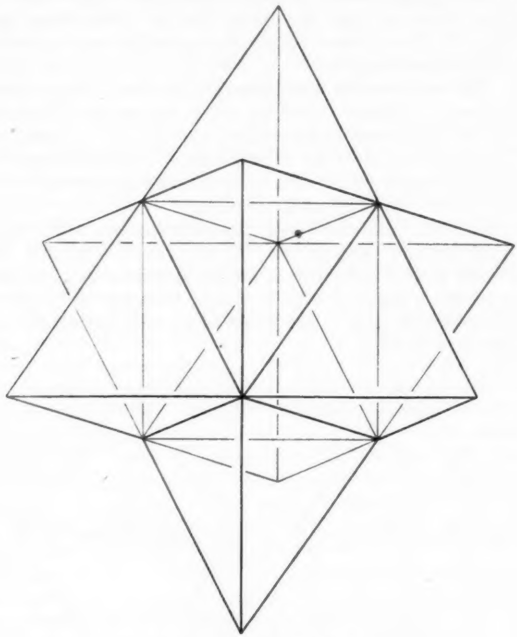


FIG. 2.—Stellated octohedron.

be interpreted as two interlaced tetrahedra, one point up and one point down. Of its 36 edges, 24, which are by twos in line with each other, form the twelve diagonals of the faces of a cube. The figure can also be formed by subtracting from this cube twelve irregular tetrahedra, each having for base a quarter of a face of the cube, for vertex the middle of the adjacent face and for volume $\frac{1}{24}$ of the cube, whence the volume of the stellated octohedron is half the volume of the cube.

ICOSAHEDRON. Along the twelve edges of an octohedron, and cyclically round each triangle, cut off equal pieces, thus obtaining twelve points which are the vertices of some sort of icosahedron, regular or irregular. 24 of its edges are necessarily equal to each other, and outline eight equilateral triangles on the faces of the octohedron: the other six are equal to each other, but not to the first 24 unless the equal pieces are chosen so as to make them so,

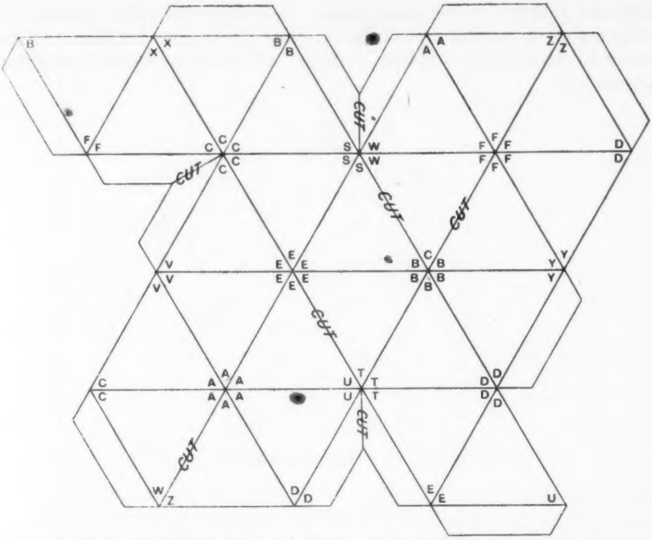


FIG. 3.—Net of stellated octohedron. *A*, *B*, *C*, *D*, *E*, *F* are the vertices of the octohedron. Fold inwards along the lines joining them, outwards along the other lines.

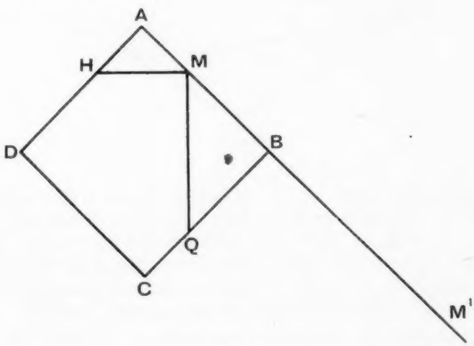


FIG. 4.

In figure 4, A, B, C and D are four vertices of an octohedron forming a square: M is the chosen point on AB. Then if the icosahedron is regular, HM is a side, and QM a diagonal, of equal regular pentagons; and these are in the extreme and mean ratio,

$$\sqrt{5}-1:2, \quad \text{or} \quad 2:\sqrt{5}+1.$$

AM and *BM* are in the same ratio: hence we have the remarkable property of a regular icosahedron, that its twelve vertices can be found by dividing in "golden section" the twelve edges of a regular octohedron.

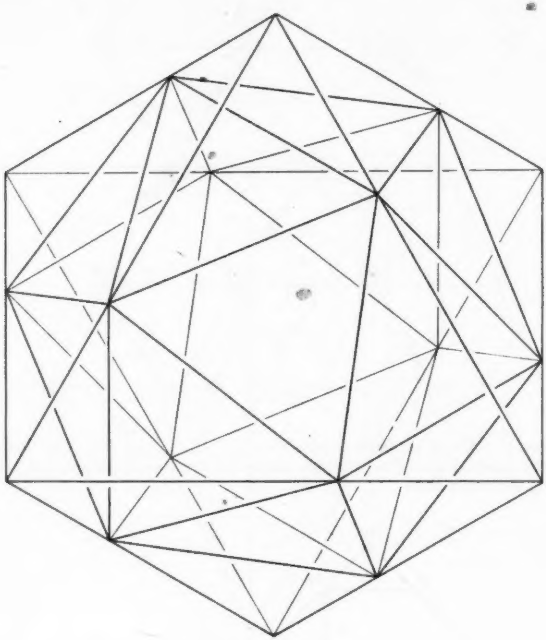


FIG. 5.

Four icosahedra can be so obtained from every octohedron, because M can be taken at a distance $(\sqrt{5}-1)/2$ of the edge, inwards from either A or B, or M' at an outward distance $(\sqrt{5}+1)/2$ of the edge. Figure 5 shows the edges of the octohedron divided internally, giving an icosahedron which is smaller, easier to imagine, and easier to make a model of than that got by the external section.

Figure 6 shows the exterior icosahedron. Points on adjacent edges of the octohedron are adjacent vertices of the interior icosahedron; but in the exterior case they are next-but-one points, and joined by pentagon-diagonals like QM.

If a vertex of the octohedron is joined to three of the four nearest vertices of the interior icosahedron, and the parallelepiped so begun is completed, its opposite corner is at another vertex of the icosahedron. (Two cases to consider.)

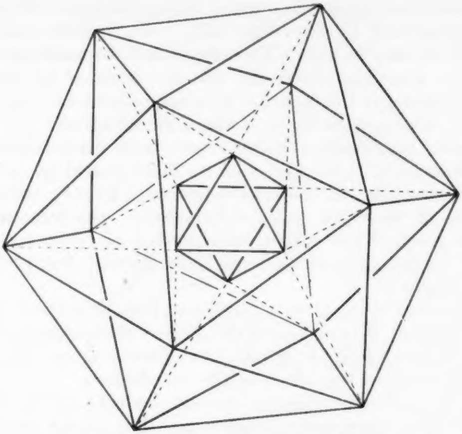


FIG. 6.—Icosahedron external to octohedron.

DODECAHEDRON. I know of no equally simple and beautiful connection between the octohedron and dodecahedron: those that follow are mere consequences of relations between the octohedron and icosahedron or between the dodecahedron and cube.

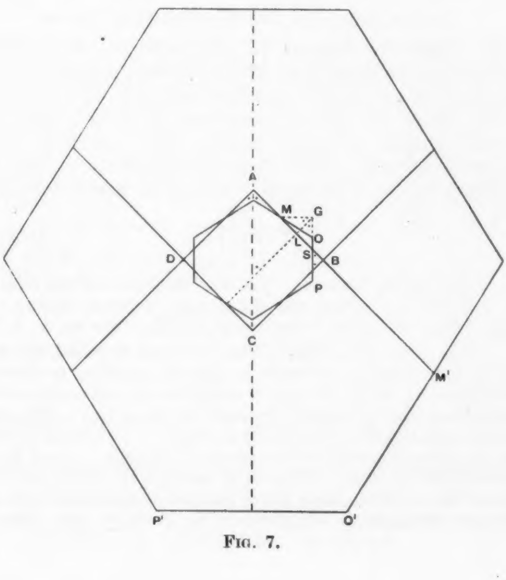


FIG. 7.

Six edges of a dodecahedron being chosen so that every pentagonal face is represented by one edge only, their middle points are the vertices of an octohedron. The choice can be made in five ways, and the five resulting octohedra can be obtained by choosing one and then rotating it through 72° at a time about an axis joining the middles of two opposite faces of the dodecahedron.

An interior icosahedron being made from an octohedron as described above, and the middles of its faces joined so as to make a dodecahedron, eight of the dodecahedron's twenty vertices are at the middles of the faces of the octohedron: the remaining twelve are all one-third of the way from a vertex of the octohedron to a vertex of the icosahedron (not the most distant, but placed as H is from B in figure 4).

Other dodecahedra have the middles of their faces at the vertices of the icosahedra. Every face of the interior dodecahedron so formed is pierced at its middle by an edge of the octohedron: six of its edges also cut two each of the edges of the octohedron.

In figure 7, AB is an edge of the octohedron, OP of the interior, and $O'P'$ of the exterior dodecahedron. The angles marked with a dot are 45°. The figure is not remarkable except for a rich crop of extreme-and-mean sections, and for the presence of seven lines in Geometrical Progression, BS, SL or LM, MA or BL, BM or LA, BA, $M'B$ and $M'A$.

W. Hope-Jones.

CORRESPONDENCE.

To the Editor of *The Mathematical Gazette*.

Sir,—*The Times* for January 10, 1942 contains the news that one of our members, the Right Rev. Walter Robert Adams, D.D., Bishop of Kootenay, has been appointed Archbishop of Kootenay. I hope that I may speak for all our members in sending him our most cordial good wishes, in congratulating the Church in Canada on having a mathematician at the helm, and the Mathematical Association on numbering (is it for the first time?) an Archbishop among its members.

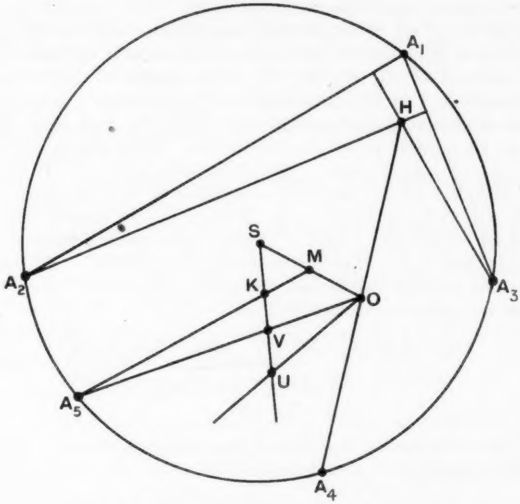
Yours, etc., W. Hope-Jones.

1391. . . . the Giralda Tower . . . was once the Tower of the Great Mosque, and was designed by Geber, the Moor who invented algebra.—Halliday Sutherland, *The Arches of the Years*, ch. ix, p. 125. [Per Mr. P. J. Harris.]

1392. . . . *The Leap of the Bells.* When the rope is pulled, the bell begins to swing and then revolves outwards or inwards according to the winding of the rope on the axle. When the rope is unwound, the bell continues to revolve by its momentum and the rope is rewound on the axle. . . . The boy stands on the parapet and throws his weight on the rope. . . . Twisting the rope round his wrists, he jerked it towards the centre of the arch . . . and the boy was slung clear out of the belfry. This flight through the air at the end of the rope towards the revolving mass thirty feet above resembled part of a parabola.—Halliday Sutherland, *The Arches of the Years*, p. 126. [Per Mr. P. J. Harris.]

MATHEMATICAL NOTES.

1566. *Cartesian coordinates and the triangle.*



In dealing with four concyclic points, it is sometimes convenient to choose for axes the asymptotes of the rectangular hyperbola through the points; the coordinates of the points are then $(t_i, 1/t_i)$, $i=1, 2, 3, 4$, so that the trigonometry involved by the use of the coordinates $(a \cos t_i, a \sin t_i)$ can be avoided. One reason for the success of this particular choice of axes for the theorem on p. 153 of the *Gazette* (July, 1941) is that the point of concurrence is the centre of the hyperbola. It is the centre of the hyperbola because the locus of the centres of conics through four orthocentric points is the nine-points circle.

The chord t_2t_3 of $x : y : 1 = t^2 : 1 : t$ is

$$x + t_2t_3y = t_2 + t_3$$

and it is perpendicular to the chord t_1t_0 if

$$t_0t_1t_2t_3 = -1.$$

The symmetry of $t_0t_1t_2t_3 = -1$ shows that t_0 is the orthocentre of $t_1t_2t_3$.

If the circle is $(x-f)^2 + (y-g)^2 = h^2$, t_1, t_2, t_3, t_4 are the roots of

$$t^2(t-f)^2 + (1-gt)^2 = h^2t^2.$$

Thus $$\Sigma(t) = 2f, \quad \Sigma(1/t) = 2g, \quad t_1t_2t_3t_4 = 1.$$

These equations determine the centre (f, g) of the circle in terms of t_1, t_2, t_3. Also $t_4=-t_0$ shows that the origin is the mid-point of the join of t_4 to the orthocentre of $t_1t_2t_3$. Hence the joins of four concyclic points to the orthocentres of the triangles formed by the remaining points bisect one another at O.

Since $HO=\frac{1}{2}HA_4$, O lies on the nine-points circle of $A_1A_2A_3$. It is the point of concurrence of the four nine-points circles.

The centroid of A_1, A_2, A_3, A_4 is the same as that of 2 at the circumcentre S, 1 at the orthocentre H, and 1 at A_4. It is therefore the same as that of 2 at S and 2 at O, and is the mid-point M of SO.

Let A_5 be a fifth point on the circle. Then the centroid K of A_1, A_2, A_3, A_4, A_5 divides A_5M in the ratio of $4:1$. If OA_5 meets SK at V, $OV=\frac{1}{2}VA_5$, and if the parallel to SA_5 through O meets SK at U, $OU/SA_5=OV/VA_5=\frac{1}{2}$. Thus O lies on the circle centre U, radius $\frac{1}{2}SA$; and $SU=\frac{5}{2}SK$. The ten nine-points circles are concurrent in fours on this circle (*Gazette*, p. 155). The relations between the various points are shown in the figure; they can also be found by using vectors as follows:

$$A_1+A_2+A_3=2S+H \quad \text{and} \quad A_4+H=2O.$$

Hence
$$2O+A_5=(A_1+A_2+A_3+A_4+A_5)-2S$$
$$=5K-2S=3V.$$

With the same axes, the join of the origin to the foot of the perpendicular from t_4 to t_2t_3 is found to be

$$x(t_1+t_2+t_3-t_4)=y(1/t_1+1/t_2+1/t_3-1/t_4)$$

and this is the Simson's Line of t_4 wo the triangle $t_1t_2t_3$.

In dealing with four points regarded as forming a triangle $A_1A_2A_3$ with a point Q on its circumcircle, there is another choice of axes which is sometimes useful. Since there is a conic, namely the circle, passing through the vertices of the triangles $A_1A_2A_3$ and QIJ, there must be another conic touching the sides of these triangles. This is a parabola with focus Q.

When axes are taken so that $X=T$, $Y=T^2$ are parametric envelope equations of this parabola, the sides of $A_1A_2A_3$ are given by values T_1, T_2, T_3 of the parameter, and Q is the point $(0, 1)$.

The line joining $(0, 1)$ to $(-1/T, 0)$ is perpendicular to the line $[T, T^2]$; hence the feet of the perpendiculars from Q to the sides of $A_1A_2A_3$ lie on the line $y=0$, which is the tangent at the vertex of the parabola.

A_1 is the point $(-1/T_2-1/T_3, 1/T_2T_3)$ and the perpendicular from A_1 to A_2A_3 is

$$y/T_1-x=1/(T_1T_2T_3)+1/T_2+1/T_3.$$

This meets the directrix $y=-1$ at the point

$$(-1/(T_1T_2T_3)-1/T_1-1/T_2-1/T_3,\ -1)$$

which, by symmetry, is the orthocentre of $A_1A_2A_3$. A. R.

1567. *Technique in Analytical Geometry.*

It is very truly remarked by Mr. E. G. Phillips (*Gazette*, pp. 144-146) that students are apt to make the mistake of starting with the ends of a chord when solving a problem that is essentially concerned with the mid-point. The example (I, p. 145) is certainly one in which the ideal solution is not usually suggested to the beginner by his common sense. But it is not only in connection with mid-points that this occurs. There are many other types of problem which the beginner starts in the wrong way. For example :

(i) Find the equation of the join of a given point to the meet of two given lines ;

(ii) Find the equation of the circle on the chord $fx+gy=1$ of $ax^2+by^2=c$ as diameter ;

(iii) P, Q are the contacts of tangents from a given point to a given conic ; the normals at P, Q meet at L ; LR, LS are the other normals from L ; the tangents at R, S meet at U ; find the coordinates of U and L.

Failure over the mid-point problem is possibly due to a weakness in technique rather than to ignorance of the equations

$$(x-x_1)/\cos\theta=(y-y_1)/\sin\theta=r, \qquad (1).$$

Or, it may be that these equations are sometimes presented in such a way that their significance is not clear. They are essentially parametric equations ; it would help if they were originally given in the form $x=x_1+r\cos\theta$, $y=y_1+r\sin\theta$ as a special case of $x=a+ct$, $y=b+dt$, and if they were not converted into the form (1). It is also likely that harm is done by the custom of confining elementary courses of analytical geometry to " the straight line and circle ". These loci do not sufficiently illustrate the value of parameters ; for, with the straight line, it is often better not to use them, and the parametric equations of a circle are not so simple as those of the parabola or hyperbola. It might be better if the elementary course was on " the straight line and the curves

$$x:y:a=t^2:t:1 \quad \text{and} \quad x:y:a=t^2:1:t."$$

Anyhow the student cannot be blamed for neglecting to use parametric equations of the line in Example I. It can be argued that it is, in fact, simpler to find the equation of the chord in another way :

Joachimsthal's equation $s_{11}k_1{}^2+2s_{12}k_1k_2+s_{22}k_2{}^2=0$ shows that the polar of P_1 goes through P_2 if $s_{12}=0$; hence the polar is $s_1=0$. The definition of the polar shows that the chord whose mid-point is P_1 is parallel to the polar of P_1 ; hence the chord is $s_1=s_{11}$.

[The notation used here differs from that on p. 144. Here s_{12} denotes $ax_1x_2+h(x_1y_2+x_2y_1)+\ldots+g(x_1+x_2)+\ldots+c$, and s_{11}, s_1, s denote the expressions obtained by changes of the suffix 2 into 1 or by omission of the suffix. Thus S_1, P_{12}, S_2 on p. 145 are replaced by s_{11}, s_{12}, s_{22} and $s_1=0$ is the polar of (x_1y_1) wo $s=0$.]

Also in Example II (p. 146) there is no advantage in the use of equations (1). The polar wo $xy = kz^2$ of the point $(x_1, y_1, 0)$ at infinity on the given line is $xy_1 + yx_1 = 0$ which is independent of k. Hence the mid-point is the meet of the given line and $xy_1 + yx_1 = 0$ for every conic with given asymptotes $xy = 0$.

Alternatively the conics form a system through four points H, H, K, K, at infinity. Therefore they form an involution on any fixed line. One double point of the involution, given by the conic HK, HK, is at infinity; hence the other double point is the mid-point of the chord for every conic of the system.

Thus the importance of equations (1) is scarcely proved by the two examples given. Perhaps its application to Newton's Theorem is more convincing. Conjugate diameters are more naturally treated as conjugate lines through the centre. In three-dimensional analytical geometry it has been customary to use the equations $(x-a)/l = (y-b)/m = (z-c)/n$, but it is often better to use coordinates which are the ratios of the determinants of the matrix

$$\begin{bmatrix} x_1 & y_1 & z_1 & t_1 \\ x_2 & y_2 & z_2 & t_2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} l & m & n & 0 \\ a & b & c & 1 \end{bmatrix}$$

A. R.

1568. *V for Victory.*

The game of chess is a very small example of plane lattice geometry, played on a very small lattice 8×8. It is hedged about by arbitrary rules, which are no doubt designed to make it a good war-game. But there are at any rate three pieces which behave in an orderly manner: the Rook, which runs along a (1, 0) or (0, 1) line; the Bishop, which runs along a (1, 1) line; and the Queen, which combines these two motions. On the other hand, the King is confined to moving one square each way, except in castling, while the Knight has only the (1, 2) or (2, 1) leap, instead of going on in a line. The Pawn moves one or two squares on the first move, capturing diagonally, with the promotion of pawns to pieces forming the climax of arbitrariness.

In Generalised Chess these varied ideas are organised. There are boards of greater size, pieces are classified into (m, n) leapers and (m, n) riders (that is, leapers continuing to leap on in a straight line), pawns are generalised into pieces which move in one way and capture in another; and so on.

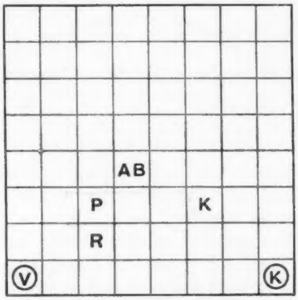
Among the family of leapers there occurs the 5-leaper, which combines in its movement the leaps from (0, 0) to (5, 0), (0, 5), (3, 4) and (4, 3), since $5^2 = 3^2 + 4^2$. This piece fits neatly into the 8×8 board. On larger boards we can have the 13-leaper, where $13^2 = 5^2 + 12^2$; and generally we may consider the $\sqrt{(m^2 + n^2)}$ leaper, where $m^2 + n^2$ is a perfect square.

If any one of these leapers makes two moves in succession (but not returning to its starting point) we get a figure V.

Placing such a 5-piece at the origin, it will be found to command ten squares on an 8×8 board, namely: (0, 6); (1, 3); (1, 7); (2, 4); (3, 1); (4, 2); (5, 5); (6, 0); (7, 1); (7, 7); these are symmetrically placed about the line $y = x$.

The calculation of the angles of the various V's forms a simple trigonometrical exercise for a matriculation form. Denoting arc tan (p/q) by (p/q), they are, in ascending order of magnitude: $(4/3) - (3/4)$ or $(7/24)$, $(3/4)$, $(4/3)$, $2(3/4)$, $(3/4) + (4/3)$ or $90°$, and $90° + 2(3/4)$; but not $2(4/3)$ since the board is not big enough.

In conclusion it is necessary to show a definite " chess " problem to illustrate these " Victory " moves.



Archbishop (3, 3); V piece (0, 0). Stalemate in 2.

(The origin is at the *centre* of the V square. Black pieces ringed.)

The Archbishop at (3, 3) adds to the power of an ordinary Bishop that of optical reflection at one edge of the board. The first move is Rook to (6, 1); and then wherever the V piece moves, as above, it can be captured by the Archbishop, the Rook or the Pawn, so that Black has no move left, it being his turn to play.

N. M. Gibbins.

1569. *A much-neglected equation.*

In the July 1941 *Gazette*, p. 144, Mr. E. G. Phillips writes of a much-neglected equation. The neglect becomes all the stranger if we notice that a beginner can obtain the equation by a simple method. For example, for the ellipse $x^2/a^2 + y^2/b^2 = 1$, the equation

$$\frac{(x-h)(x-h')}{a^2} + \frac{(y-k)(y-k')}{b^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

represents the chord joining (h, k) and (h', k') because it is of the first degree and passes through these two points if they are on the curve. The slope of this line

$$= -\frac{b^2}{a^2} \cdot \frac{h+h'}{k+k'}$$

$$= -\frac{b^2}{a^2} \cdot \frac{x_1}{y_1} \text{ if } (x_1, y_1) \text{ is the mid-point.}$$

Hence the equation to the chord in terms of its mid-point is

$$y - y_1 = -(b^2x_1/a^2y_1)(x - x_1)$$

or

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$$

in its usual form. Textbooks as a rule ignore this.

For the general equation of the second degree, the equation to the chord being

$$A(x-h)(x-h') + H\{(x-h)(y-k') + (x-h')(y-k)\} + B(y-k)(y-k')$$
$$= Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C,$$

its slope would be

$$\{A(h+h') + H(k+k') + 2G\}/\{B(k+k') + H(h+h') + 2F\}$$
$$= \{Ax_1 + Hy_1 + G\}/\{By_1 + Hx_1 + F\}.$$

W. ALDERSEY LEWIS.

1570. *Note on an entry in Lewis Carroll's diary.*

In *The Life and Letters of Lewis Carroll* written in 1898 by Stuart Dodgson Collingwood, there occur the following entries from his diary for 1890 when he was 58.

"Oct. 31st. This morning, thinking over the problem of finding two squares whose sum is a square, I chanced on a theorem (which seems *true*, though I cannot prove it), that if $x^2 + y^2$ be even, its half is the sum of two squares. A kindred theorem, that $2(x^2 + y^2)$ is always the sum of two squares, also seems true and unprovable."

"Nov. 5th. I have now proved the above two theorems. Another pretty deduction from the theory of square numbers is, that any number whose square is the sum of two squares, is itself the sum of two squares."

The solution of his two problems of Oct. 31st is really trivial. For the first, if $x^2 + y^2$ be even and x, y are both integers, then x, y are both odd or both even, and so $x \pm y$ are both even. Hence

$$\tfrac{1}{2}(x^2 + y^2) = \{\tfrac{1}{2}(x+y)\}^2 + \{\tfrac{1}{2}(x-y)\}^2.$$

For the second,

$$2(x^2 + y^2) = (x+y)^2 + (x-y)^2.$$

It seems exceedingly difficult to take his remarks at their face value. I have sometimes wondered over them, and also entertained guests by quoting them. Perhaps some reader may be able to elucidate this mystery.

For the sake of completeness, I remark that the entry of Nov. 5th presumably means that if $x^2 = a^2 + b^2$ where x, a, b are integers, then $x = c^2 + d^2$ where c, d are integers. The statement, however, may not be true for all x if either a or b is zero. It is merely a way of stating part of the solution of the well-known Pythagorean equation $x^2 = y^2 + z^2$ and has been known for nearly a thousand years.

L. J. MORDELL.

1571. *The unit crescent.*

Though perhaps our pupils do not appreciate as much as we hope the distinction between the two cases, I am sure we all enjoy pointing out to them that the area of a parabolic segment, unlike the area of a circle, is commensurable with the rectangle which contains it.

Perhaps however it might be well to lay more stress than is usual on commensurable areas bounded by circular arcs.

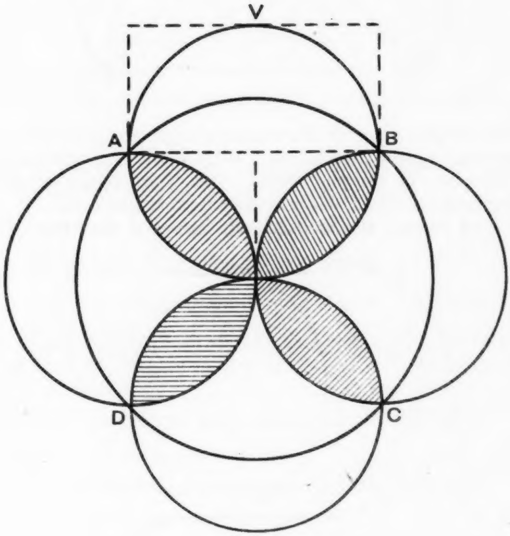


Fig. 1.

Fig. (1) shows one of the simplest cases of such areas.

If $ABCD$ is a square of side two inches each of the unshaded areas, whose shape is like that of a fat " ice-cream-cone ", is precisely two square inches.

Further the larger circle exactly bisects each of these areas, so that, except for the shaded areas, the figure—if drawn so that AB is 2″—contains eight areas bounded by circular arcs each of which is exactly a square inch.

The two-square-inches area is easily converted into two square inches by cutting away the part below AB, cutting it in two and placing the parts in the top left and top right corners of the dotted rectangle.

That the larger circle bisects this area, so forming the unit crescent and the unit circular triangle, is less obvious but can be seen from Fig. (2). This is the well-known adaptation of the Euclidean theorem about similar figures described on the sides of a right-angled

triangle, which shows that two such crescents are together equal to the triangle itself—since the two smaller semicircles are together

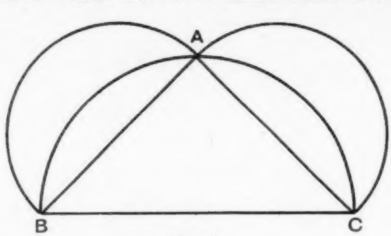
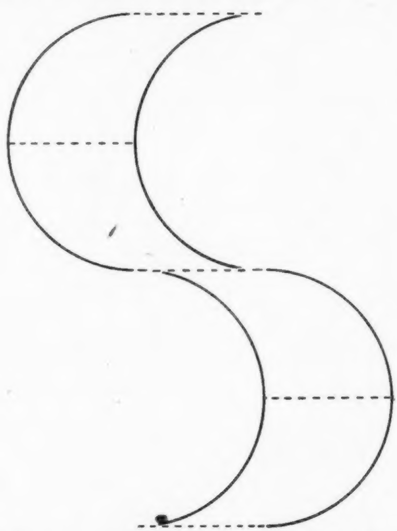


FIG. 2.

equal to the larger one and the common segments may be removed.

Two queries suggest themselves. Why is the crescent moon never in the shape of the unit crescent? Are there many other simply shaped commensurable areas bounded by circular arcs?

There is, of course, the well-known figure of the 'Snaky Square



Inches', involving the question "what is the snake's least thickness"?

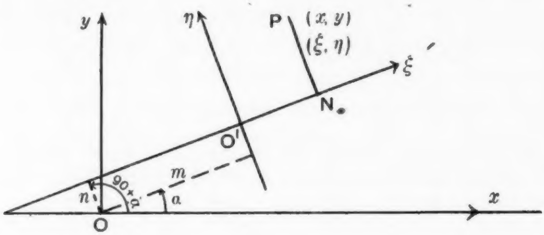
C. O. TUCKEY.

1572. *Transformations of Cartesian equations.*

In Note 1531 (*Gazette*, XXV, July 1941, p. 175), Dr. Maxwell suggests the use of the transformation equations

$$\rho\xi = px + qy + a, \quad \rho\eta = py - qx + b,$$

(correcting an obvious misprint in the second equation). May I suggest the following simple methods of obtaining these two equa-



tions and at the same time of determining ρ. Let the equation of the new η-axis, referred to the old axes Ox, Oy be

$$x \cos \alpha + y \sin \alpha - m = 0,$$

then that of the new ξ-axis referred to Ox, Oy is

$$-x \sin \alpha + y \cos \alpha - n = 0.$$

Let P be any point whose coordinates referred to the original axes are (x, y) and to the new axes (ξ, η). Draw PN perpendicular to $O'\xi$.

Then $\qquad \xi =$ length of perpendicular from P to $O'\eta$

$$= x \cos \alpha + y \sin \alpha - m,$$

and $\qquad \eta =$ length of perpendicular from P to $O'\xi$

$$= -x \sin \alpha + y \cos \alpha - n.$$

Let $\quad \cos \alpha = p/\surd(p^2+q^2)$; then $\sin \alpha = q/\surd(p^2+q^2)$.

We then have

$$\xi\surd(p^2+q^2) = px + qy - m\surd(p^2+q^2), \quad \text{..................(i)}$$

and

$$\eta\surd(p^2+q^2) = -qx + py - n\surd(p^2+q^2). \quad \text{...............(ii)}$$

Put $\surd(p^2+q^2) = \rho$; $\quad m\surd(p^2+q^2) = -a$; $\quad n\surd(p^2+q^2) = -b$.

Equations (i) and (ii) become

$$\rho\xi = px + qy + a,$$

$$\rho\eta = -qx + py + b, \quad \text{where} \quad \rho = \surd(p^2+q^2).$$

A. G. CARPENTER.

1573. *An analytical proof of Pascal's theorem.*

1. Professor Watson's Note (No. 1511) prompts me to suggest the following as a rather simpler solution.

Let the six-point be $AFBDCE$, with ABC as triangle of reference and $D(x_1, y_1, z_1)$, $E(x_2, y_2, z_2)$, $F(x_3, y_3, z_3)$.

The equation of the conic is $a/x+b/y+c/z=0$ with the condition

$$\Delta=\begin{vmatrix} \dfrac{1}{x_1}, & \dfrac{1}{y_1}, & \dfrac{1}{z_1} \\ \dfrac{1}{x_2}, & \dfrac{1}{y_2}, & \dfrac{1}{z_2} \\ \dfrac{1}{x_3}, & \dfrac{1}{y_3}, & \dfrac{1}{z_3} \end{vmatrix}=0, \quad \text{and} \quad \begin{array}{l} \dfrac{a}{X_1}=\dfrac{b}{Y_1}=\dfrac{c}{Z_1}, \\ \dfrac{a}{X_2}=\dfrac{b}{Y_2}=\dfrac{c}{Z_2}, \\ \dfrac{a}{X_3}=\dfrac{b}{Y_3}=\dfrac{c}{Z_3}, \end{array}$$

where X_1, Y_1, etc., are the minors of $1/x_1$, $1/y_1$, etc., in Δ.

AF is $y/y_3=z/z_3$, CD is $x/x_1=y/y_1$ and they meet at

$$(x_1y_3,\ y_1y_3,\ y_1z_3).$$

(AF, CD), (BD, AE) and (CE, BF) are collinear if

$$\begin{vmatrix} x_1y_3, & y_1y_3, & y_1z_3 \\ x_2x_3, & y_2x_3, & z_3x_2 \\ x_1z_2, & y_2z_1, & z_1z_2 \end{vmatrix}=0.$$

Dividing the rows by y_1y_3, x_2x_3, z_1z_2 and the columns by x_1, y_2, z_3, the condition reduces to $\Delta=0$.

The points are therefore collinear.

If the equation of the Pascal line is $lx+my+nz=0$, then, using the conditions that the coordinates of the second and third of the points of intersection satisfy this equation, $\dfrac{lx_1}{Y_1}=\dfrac{my_2}{Y_2}=\dfrac{nz_3}{Y_3}$, and the line is $P_1\dfrac{x}{x_1}+P_2\dfrac{y}{y_2}+P_3\dfrac{z}{z_3}=0$, where P is X, Y or Z.

The equation may also be written

$$\frac{xX_1}{ax_1}+\frac{yY_2}{by_2}+\frac{zZ_3}{cz_3}=0.$$

2. Professor Watson's analytical proof suggests an alternative and more symmetrical method of approach. The notation used is that of Note 1511. If the points (12, 45), etc., are collinear let the equation of the line joining them be $lx-my+nz=0$.

Then
$$\begin{vmatrix} l, & m, & n \\ t_1t_2, & t_1+t_2, & 1 \\ t_4t_5, & t_4+t_5, & 1 \end{vmatrix}=0,$$

i.e. $l\{(t_1-t_4)-(t_5-t_2)\}+m\{t_4t_5-t_1t_2\}$

$$+n\{t_2t_5(t_1-t_4)-t_1t_4(t_5-t_2)\}=0, \quad \text{............(1)}$$

with similar equations (2) and (3) formed by changing t_1 into t_3, t_2 into t_4, etc., for (2) and the same cyclical change again to obtain (3) from (2).

That (1), (2) and (3) can be simultaneously satisfied by one set of values of l, m, n is shown by multiplying (1) by (t_3-t_6), (2) by (t_5-t_2) and (3) by (t_1-t_4) and adding the equations, leading to an

identity. The points are therefore collinear. l, m, n are then found by taking $(1)+(2)+(3)$ and $(1)\times t_3t_6+(2)\times t_2t_5+(3)\times t_1t_4$, which lead to $l/C=m/B=n/A$, where

$$A=\{(t_1t_2+t_3t_4+t_5t_6)-(t_4t_5+t_6t_1+t_2t_3)\},$$

$$B=(t_1-t_4)(t_2t_5-t_3t_6)+(t_3-t_6)(t_1t_4-t_2t_5)+(t_5-t_2)(t_3t_6-t_1t_4),$$

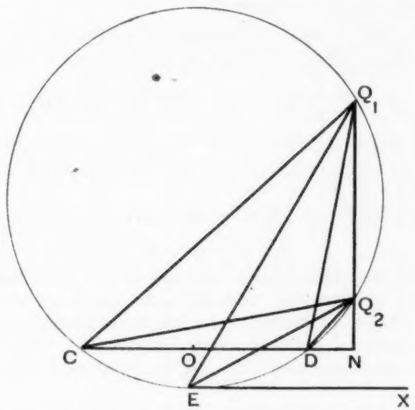
$$C=\{(t_3t_4t_5t_6+t_5t_6t_1t_2+t_1t_2t_3t_4)-(t_6t_1t_2t_3+t_2t_3t_4t_5+t_4t_5t_6t_1)\}.$$

As A, B and C are irreducible the number of Pascal lines which can be formed from six given points on a conic is equal to half the number of *distinct* values which one of them can assume on permutation of the suffixes, for there is one permutation of any A which gives $-A$ and this also changes B into $-B$ and C into $-C$. By keeping t_1 fixed and permuting the remainder it is at once seen that there are sixty such Pascal lines. S. T. SHOVELTON.

1574. *A problem in elementary geometry.*

The paper "On a Problem in Elementary Geometry" (*Gazette*, No. 265, July 1941, p. 136) is interesting for its analysis but the problem may be solved more directly by a purely geometrical method.

Draw the triangle CQD in the plane of ABP, CD coinciding with AB, and draw its circumcircle. O is the mid-piont of CD, EX the



tangent parallel to CD and N is the projection of Q on CD. Q is at Q_1 or Q_2 and P is between Q and N. Thus the condition that the orthogonal projection of $\angle CQD$ on a plane through CD should be less than $\angle CQD$ is that P should be between Q_2 and N.

In the notation of the paper $NO=y$, $OD=OC=d$, $NP=x$, $NQ=\surd(x^2+h^2)$, and γ is the acute angle between QE and CD. Denote the two possible values of γ by γ_1 and γ_2.

Then, as $\gamma_2 = \angle XEQ_2 = \angle EQ_1N$, it follows that $\gamma_1 + \gamma_2 = \frac{1}{2}\pi$, and therefore that $\gamma_2 \not> \frac{1}{4}\pi$. Thus if $\gamma \not> \frac{1}{4}\pi$, Q must be at Q_2 and P must necessarily be outside the circle. Algebraically the condition may be written $x^2 + h^2 = NQ_2{}^2 \not> NQ_1 \,.\, NQ_2 \not> y^2 - d^2$. If, however, $\gamma > \frac{1}{4}\pi$, Q is at Q_1 and the condition is that

$$x < NQ_2, \quad \text{or} \quad x\sqrt{(x^2+h^2)} < NQ_1 \,.\, NQ_2 < y^2 - d^2.$$

S. T. SHOVELTON.

1575. *On Note* 1530.

The determination of the curvature of $x^3 + y^3 = 3axy$ at the origin is an excellent example of what I have called (*Gazette*, vol. xxiv, p. 277) delayed action. When $x=0$, $y=0$, the equation

$$x^2 + y^2 \frac{dy}{dx} = ay + ax \frac{dy}{dx}$$

teaches us nothing, but if we differentiate, we have

$$2x + 2y\left(\frac{dy}{dx}\right)^2 + y^2 \frac{d^2y}{dx^2} = 2a\frac{dy}{dx} + ax\frac{d^2y}{dx^2},$$

and now we learn that if finite derivatives are to exist the only possible value of dy/dx is 0. Differentiating again, we have an equation which, for $x=0$, $y=0$, $dy/dx=0$, gives us d^2y/dx^2 unhampered by d^3y/dx^3.

E. H. N.

1576. *A property of the parabola.*

If PSQ is a focal chord of a parabola, vertex A, the equation to the circle on PQ as diameter may be written in the form

$$(1) \qquad (x - at^2)\left(x - \frac{a}{t^2}\right) + (y - 2at)\left(y + \frac{2a}{t}\right) = 0.$$

The power of the point A w.r. to this circle is obtained by substituting $(0, 0)$ in the left-hand side of this equation, and therefore equals $-3a^2$.

Inverting the circle on PQ as diameter w.r. to the circle,

$$x^2 + y^2 + 3a^2 = 0,$$

the circle will invert into itself. The line $x + a = 0$, which it touches, is transformed by this inversion into the circle on the join of the points $(0, 0)$, $(0, 3a)$ as diameter. This circle therefore also touches (1), and the point of contact must be collinear with the point of contact of the directrix and (1), and the centre of inversion, which is the vertex A.

The centres of the circles (1) lie on the parabola

$$y^2 = 2a(x - a),$$

and, being always orthogonal to a fixed circle, these circles form a *conic system* * of circles. The envelope of a general conic system of circles is a bicircular quartic curve.* In this case the envelope degenerates into a line, a circle, and the line at infinity.

* *Cf.* "On the Representation of Circles by Means of Points in Space of Three Dimensions." *Math. Gazette*, vol. 21, July 1937.

These remarks may serve to amplify Note 1522, in which Mr. E. P. Lewis obtained the envelope property of the circles (1) by pure geometry.

D. PEDOE.

1577. *Motion with changing mass.*

In a recent article,* Mr. A. S. Ramsey drew attention to the much-neglected problem of the motion of a body whose mass is changing continuously and suggested a general procedure for the solution of such problems. The object of the present note is to indicate an alternative procedure for some simple problems of this type, and to stress the importance of a general equation implicit in Ramsey's work (and actually given in Palmer and Snell's *Mechanics*, p. 290, equation (2)).†

Consider the motion of a body in the presence of a number of material particles, the continuous variation of the mass of the body arising from the absorption or ejection of such particles. Now instead of considering the motion of the body itself, consider the motion of the body and particles as a single mechanical system. For this system, at any instant, the time rate of change of linear momentum is equal to the resultant force acting on the system, and this statement may readily be expressed as an equation of motion in any particular case for which the external forces acting on the particles may be neglected. In Ramsey's example (i), the motion of the water, once it has left the engine, obviously will not affect the motion of the engine, and we may regard the horizontal motion of the system at time t as that of an engine of mass M moving with speed V together with a number of water drops moving with various constant horizontal components of velocity. For the system as a whole, the horizontal component of the external force is equal to the time rate of change of the momentum of the engine plus the rate at which the set of water drops is gaining horizontal momentum. This gives at once Ramsey's equation at the top of page 143.

The above procedure may be used to establish an important general equation of motion. At time t let the body have mass M, vector velocity $\mathbf{V}$ and be acted upon by a force $\mathbf{F}$. Suppose that at this time the linear momentum vector of those particles not attached to the body is $\mathbf{p}$. Considering the case for which there are no external forces acting on the particles, we have for the whole system

$$\frac{d}{dt}\{M\mathbf{V}+\mathbf{p}\}=\mathbf{F}. \qquad \text{.............................(1)}$$

Now if the mass of the body increases by δM in time δt, the mass of the set of particles will decrease by this amount, and the vector momentum lost by the particles will be $\mathbf{u}\,\delta M$, where $\mathbf{u}$ is the velocity at time t of the mass centre of those particles absorbed by the body in time δt. Similarly if the body loses mass δM, the momentum gained by the

* *Gazette*, July 1941, p. 141.

† My attention was drawn to this fact by Mr. E. G. Phillips.

particles is $\mathbf{u}\,\delta M$, where $\mathbf{u}$ is now the velocity at time $t+\delta t$ of the mass centre of those particles ejected by the body in time δt. In either case $-d\mathbf{p}/dt=\mathbf{u}dM/dt$, and equation (1) becomes

$$\frac{d}{dt}(M\mathbf{V})-\mathbf{u}\frac{dM}{dt}=\mathbf{F}. \qquad \text{.........................(2)}$$

This equation * is actually of quite general validity, for the important factor is the velocity of matter at the instant of absorption (or ejection); its past (or future) behaviour will not in general affect the motion of the body. Equation (2) reduces to Ramsey's equations in his examples (i) and (ii). In example (iii), a mass m of fuel is converted into a mass m' of smoke and gas in unit time, and if it be supposed that the remaining mass $m-m'$ of ash moves with the engine, $dM/dt=-m'$ (not $-m$), and the horizontal component of equation (2) gives Ramsey's equation (5). If it is supposed that the residue of combustion $(m-m')\delta t$ in time δt be ejected so that its velocity is zero, while $m'\delta t$ is still ejected as previously, then $dM/dt=-m$, but in this case $u=m'V/m$ and equation (2) again gives Ramsey's result.

A detailed proof of the general validity of equation (2) may be obtained by adopting the fundamental procedure recommended by Ramsey. Consider the case of absorption, and let the resultant force acting on the body and the element of mass δM at time t be $\mathbf{F}+\mathbf{F}'$; the resultant force at time $t+\delta t$ will be $\mathbf{F}+\delta\mathbf{F}$, since any external force $\mathbf{F}'$ acting on δM is now included in $\mathbf{F}+\delta\mathbf{F}$. We then have approximately $(\mathbf{F}+\mathbf{F}')\delta t=(M+\delta M)(\mathbf{V}+\delta\mathbf{V})-M\mathbf{V}-\mathbf{u}\,\delta M$, and, since in general $\mathbf{F}'$ will be of the same order of magnitude as δM, equation (2) will follow. The case of the ejection of matter from a body may be discussed in a similar manner.

Equation (2), which is the general equation of motion for a body with continuously changing mass, replaces the ordinary Newtonian equations for the motion of a body of constant mass. It reduces to $Md\mathbf{V}/dt=\mathbf{F}$ when $\mathbf{u}=\mathbf{V}$, and to $d(M\mathbf{V})/dt$ when $\mathbf{u}=0$, so that *the first form of Newton's equation is valid when the velocity of the matter immediately before absorption, or after ejection, is equal to the instantaneous velocity of the body, and, the second form is valid when the matter is at rest immediately before absorption, or after ejection.* In all other cases equation (2) must be used. R. N.

1578. *Linear differential equations of the second order.*

For several years I have taught the following method of solution, by means of a Riccati equation; the substance of the method is to be found in Forsyth's *Differential Equations*, but the exposition does not make clear—in fact it fails to mention—the underlying principle.

In effect, given an equation $y''+Py'+Qy=R$, we try to construct its first integral. This will be of the form $y'=\mu y+\nu$, where the functions μ and ν are to be determined, the latter containing an

* *Cf.* Palmer and Snell, *loc. cit.*

arbitrary constant of integration. Differentiating the first integral we have

$$\begin{aligned} y'' &= \mu y' + \mu' y + \nu' \\ &= (\mu' + \mu^2) y + \nu' + \mu\nu. \end{aligned}$$

Substituting in the given equation, we thus obtain

$$(\mu' + \mu^2 + P\mu + Q) y + \nu' + \mu\nu = R.$$

It is easily shown that this necessarily leads to the two equations:

$$\mu' + \mu^2 + P\mu + Q = 0,$$
$$\nu' + \mu\nu - R = 0.$$

The former is a Riccati equation ; if any particular solution to it can be found and is substituted in the latter, the function ν can then be determined, and y itself follows from the equation

$$y' = \mu y + \nu.$$

Example. $y'' + y = R$. Here we have $\mu' + \mu^2 + 1 = 0$, $\nu' + \mu\nu = R$. The first equation is satisfied by $\mu = -\tan x$, whence the second gives $\nu \cos x = \int R \cos x \, dx + C$. Finally we have thus to integrate the equation $y' + y \tan x = \sec x \int R \cos x \, dx + C \sec x$. L. Roth.

1579. *The ellipse in Nature.*

While working out an elementary course in geometry based on Nature it occurred to me to test the shape of the leaf of the Marsh-marigold or King-cup. The first three leaves tried proved to be almost perfectly elliptical.

In the first case I draw the outline of the leaf (which had shrunk overnight), then the minor and major axis, through a centre where the stalk joined the leaf. I found the potential foci and finally tested a couple of points on the outline. The result was encouraging, so I drew an ellipse with axes of the same length and placed the leaf upon it. The agreement was surprisingly good.

In the second case the outline of the leaf was drawn before it shrank, and the ellipse was drawn with the same axes as this outline. The two curves coincided except at three short sections.

The collection of leaves and comparison with geometrical shapes would probably prove a stimulating exercise for children.

R. S. Williamson.

1580. *On Note* 1522.

The final interesting result is a particular case of the property that circles whose diameters are any two chords of a parabola intersecting on the axis have their radical axis passing through the vertex. See, *e.g.*, Loney, *Elements of coordinate geometry*, 1931, p. 205, No. 24.

T. R. Dawson.

1581. *Cycles.*

By a cycle of t is meant a difference relation which gives $u_{n+t}=u_n$ for all n. Here are some cycles :

$$u_{n+1}\,u_{n-1}+p=-u_n(u_{n-1}+u_{n+1}), \qquad \text{a 3-cycle ;}$$
$$u_{n+1}\,u_{n-1}+p=u_n(u_{n-1}+u_n+u_{n+1}), \qquad \text{a 4-cycle ;}$$
$$u_{n+1}\,u_{n-1}+p=u_n(u_{n-1}+2u_n+u_{n+1}), \qquad \text{a 6-cycle ;}$$
$$u_{n+1}\,u_{n-1}-a^2=au_n, \qquad \text{a 5-cycle.}$$

The last is not of the same type as the others. Can anyone produce an interesting 7-cycle? R. C. LYNESS.

1582. *It* DOES *happen!*

My sympathies are, in general, very much with M. M. R. in her views on the De-bunking of Arithmetic expressed in No. 266 of the *Mathematical Gazette.*

Generalisations are, however, always a bit risky—and I should like to point out that, in certain very practical computations, " it *does* happen " quite often!

Compare M. M. R.'s first illustrative sum with the working shown below: this working is only part of a page from a *Navigating Officer's Work Book* after the said officer has taken an observation of the sun for the purpose of finding his ship's position. " Sums " of this type may appear as often as ten times in the *normal* day's work of a navigating officer. I have attached the " labels " which are commonly used in the Royal Navy.

D.W.T.S. -		4h	34m	50s	
		4	35	40	
		4	36	09	
	3)	13	46	39	
Mean		4	35	33	
Error -			6	50	fast
G.M.T. -		16	28	43	
Long. -		9	41	07	W.
L.M.T. -		6	47	36	
E. - -		11	53	40	
H.A.T.S. -		18h	41m	16s	

G. A. C.

1583. *All roads lead to Rome.*

From $8^{-\frac{5}{3}}\times 4^{\frac{5}{2}}$ several recognised roads lead to Rome (in this case unity), but a recent examination added these byways :

(i) $2^{-5}\times 2^{5}=4^{5-5}=4^{0}$.

(ii) $\dfrac{8^3}{8^5}\times\dfrac{4^5}{4^2}=\dfrac{4^3}{8^2}=\dfrac{64}{64}$.

(iii) $(4^{\frac{1}{2}})^5 \div (8^{\frac{1}{3}})^5 = 4^{\frac{1}{2}} \div 8^{\frac{1}{3}} = 2 \div 2$.

(iv) $5\sqrt{4} \div 5\sqrt[3]{8} = 10/10$.

(v) $\frac{5}{2} \log 4 / \frac{5}{3} \log 8 = 5 \log 2 / 5 \log 2$.

C. O. Tuckey.

1584. *Approximations to roots.*

If a is an approximation to $\sqrt{N}$, a much closer approximation is given by

$$\sqrt{N} = a + \frac{4a(N-a^2)(N+a^2)}{4a^2(N-a^2)+(N+3a^2)^2}.$$

If $N=(a+h)^2$, the error is $h^5/16N^2$ approximately. For example, with $N=10$, $a=3$ we have

$$\sqrt{10} = 3 \cdot 1622776 ;$$

with $N=274969$ and $a=524$, we can obtain the next 14 significant figures of $\sqrt{N}$.

For the fourth root, a similar formula is

$$\sqrt[4]{N} = a + \frac{4a(N-a^4)}{N+5a^4+10a^2\sqrt{N}}.$$

From this, the above formula for $\sqrt{N}$ is obtained by writing $N=N^2$.

If a is an approximation to the nth root of N, then a better approximation than the well-known Hutton formula and one which is in some ways easier to handle is given by

$$\frac{N+(n-1)a^{n/2}N^{1/2}}{a^{n-1}+(n-1)a^{(n-2)/2}N^{1/2}}.$$

R. H. Birch.

1393. "Joe bought a roll, and reduced his purse to the condition (with a difference) of that celebrated purse of Fortunatus, which, whatever were its favoured owner's necessities, had one unvarying amount in it. In these real times, when all the Fairies are dead and buried, there are still a great many purses which possess that quality. The sum-total they contain is expressed in arithmetic by a circle, and whether it be added to or multiplied by its own amount, the result of the problem is more easily stated than any known in figures."—Charles Dickens, *Barnaby Rudge*, Chapter XXXI. [Per Mr. T. R. Dawson.]

1394. I prefer talking with a coloured maid to talking with a mathematician ; her words are more concrete, her laughter is more energetic, and I generally gain more in knowledge of human nature by talking with her. I am such a materialist that at any time I would prefer pork to poetry, and would waive a piece of philosophy for a piece of fillet, brown and crisp and garnished with good sauce.—Liu Yutang, *The Importance of Living*, p. 147. [Per Mr. D. J. Finney.]

1395. Thirty thousand pigeons were released, filling the air with the flutter of a million wings.—From a commentary on a news film, quoted by R. W. Jepson in *Clear Thinking*. [Per Mr. F. J. Wood.]

REVIEWS.

Operational methods in applied mathematics. By H. S. Carslaw and J. C. Jaeger. Pp. xvi, 264. 17s. 6d. 1941. (Oxford)

The number of books on various forms and extensions of the Heaviside operational calculus is rapidly increasing. But many of these are written specially for electrical engineers, ignoring other fields of application, and frequently dismissing the fundamental theory in a manner unsatisfactory to the mathematician. On the other hand, books dealing, wholly or in part, with the Laplace integral as the basis of the operational method, such as H. T. Davis, *Theory of Linear Operators*, or G. Doetsch, *Theorie und Anwendung der Laplace-Transformation*, treat the subject with a mathematical rigour which goes above the head of the average technician looking for a tool with which to tackle, say, problems of circuit and cable analysis. The aim of the present authors is to blend theory and practice in a manner likely to satisfy both classes of student.

The main lines of their approach should be familiar to readers of the *Gazette* through the papers of Professor Carslaw and Dr. Jaeger in Nos. 250, 253, and 258. The first three chapters discuss the Laplace integral in its application to linear differential equations with constant coefficients, taking account of initial conditions, and the solution in this way of problems in dynamics and simple circuit analysis. Stated briefly, and taking a trivial example to exemplify the method without regard at the moment to points of rigour, suppose we are to find that solution of $dy/dt+y=1$ for which $y=2$ when $t=0$. We multiply by $\exp(-pt)$ and integrate from 0 to ∞. Integrating the first term by parts and denoting $\int_0^\infty y\exp(-pt)dt$ by y^*, we have †

$$-2+(p+1)y^*=1/p,$$

or, dividing by $(p+1)$ and putting into partial fractions,

$$y^*=\frac{1}{p}+\frac{1}{p+1}.$$

To return to y we have only to identify the right-hand side with known integrals ; since

$$\int_0^\infty \exp(-pt)dt=1/p$$

and

$$\int_0^\infty \exp(-pt).\exp(-kt)dt=1/(p+k), \qquad \text{......................(i)}$$

we obtain at once the required solution, namely,

$$y=1+\exp(-t).$$

In this way, once having established such results as

$$\int_0^\infty \frac{dy}{dt}e^{-pt}dt=-y_0+py^*,$$

and having collected a few simple operational forms, most of which can readily be deduced from the fundamental result (i), the solution of linear differential equations with constant coefficients is a matter of simple algebra, or, as

† The authors use $\bar{y}(p)$ for the integral here denoted by $y^*(p)$. The use of a bar might be misleading : but some general agreement on a standardised symbol would be desirable.

Heaviside would say, the problem has been " algebrized ". This particular process appears to have been first explicitly carried out by Bateman, *Trans. Camb. Phil. Soc.*, 1909. The theoretical obstacles are very evident ; we have made assumptions about the existence of limits and of integrals, and about the deduction of the form of y from the form of y^*. The latter point is met by Lerch's theorem, proved in Appendix I of the present book. For the former, at first the authors are content to verify the correctness of the solutions, but in Chapter IV we pass to more recondite considerations, where the inversion theorem, now usually called the Fourier-Mellin theorem, is used to show how the Laplace integral is connected with the contour integral, first given by Bromwich in his now classical paper of 1916 in *Proc. London Math. Soc.* This work also prepares the way for Chapter V, where the study of linear partial differential equations is begun.

The remaining chapters VI-X contain applications to the conduction of heat, vibrations of continuous systems, hydrodynamics, transmission lines and wave theory.

The volume ends with a number of appendices, one being mentioned above ; among the rest, there is a short note on Bessel functions, and a useful brief table of operational forms, or Laplace transforms. The novice should notice that the Heaviside forms contain a factor p, not present in the Laplace forms ; this arises from Heaviside's version

$$\exp(-kt) = p/(p+k)$$

of the fundamental form (i).

Besides the very large number of illustrative examples and problems worked out in full in the text, there is a generous supply for the student himself, about 150 in all, covering all the main points. In the worked problems on circuit theory, there is a point which might well puzzle the beginner. In the problem on p. 36, concerning perfectly-coupled circuits, the initial currents are stated to be zero, but the values of the currents at time t provided by the process do not vanish for $t=0$. A similar paradox arises in the problem on p. 32, and there a reference is given to Appendix III, which deals with impulsive functions, but the treatment is hardly adequate and a more convincing explanation would be welcome. The situation is one of common occurrence in circuit theory, yet most textbooks leave it obscure ; I am sorry that the present authors have not attempted to clear it up.

A word, not of adverse criticism, but of explanation, might be spared here for the title. The adjective " operational " is used in a significance historic rather than descriptive. The all-pervading p is not the d/dt which it might—or might not—be in Heaviside's superb conjuring tricks, nor the definite integral operator

$$\int_0^t (\quad)\,dt$$

of Jeffreys' effective exposition. It is a new variable, a parameter of transformation, and the whole process is one of correspondences between $y(t)$ and $y^*(p)$. There is both loss and gain ; we seem to lose something of the vivid, concrete idea of an operation ; on the other hand we are less likely to hear the despairing cry of readers of the so-called " practical " texts, " When is an operator *not* an operator? "

The authors have given us a very welcome volume on a most interesting and useful subject. The Oxford Press has shown us that even in war time it can deal as efficiently as ever with a complicated piece of type-setting.

T. A. A. B.

Sir Thomas Little Heath, K.C.B., K.C.V.O., F.R.S. 1861-1940. By M. F. Headlam. Pp. 16. 2s. 1941. From the *Proceedings of the British Academy.* (Oxford)

This reprint, with its frontispiece a noble photograph of Heath, is an appropriate tribute to one who was " a typical example of all that was best in the British Civil Service ", " intellectually . . . distinguished even among colleagues of much intellectual distinction ". It incorporates some remarks on Heath's contributions to mathematics furnished by Dr. J. Gilbart Smyly, of Trinity College, Dublin. These comments deal briefly but on the whole adequately with the great contribution Heath made to the study of Greek mathematics. But I am reluctant to accept the statement that " Heath hoped that his work . . . might bring about a return to Euclid so that he might be restored to the place which he had occupied for two thousand years ". It is, I believe, more nearly true to say that Heath did not wish the modern, and justifiable, revolt from Euclid as a textbook for young children to result in an undervaluation, by competent adults, of the extraordinary skill of the Greek geometers. T. A. A. B.

Edmond Halley as physical geographer and the story of his charts. By S. Chapman. Pp. 15, with six plates. 2s. 6d. 1941. Reprinted from *Occasional Notes*, No. 9. (Royal Astronomical Society, University Observatory, Oxford)

Perhaps the seventeenth century was the " century of genius " because the word " unilateral " had not been coined. The striking many-sidedness of its great men may be due to the smaller area of the field of science, but the student of the history of science in this period is being constantly reminded, or informed, of their Argus-like sight and their Briareus-like activity. Few have not heard of Halley's comet ; mathematicians will not readily forget his share in the publication of Newton's *Principia*, or his edition of Apollonius. In the present reprint, Professor Chapman, fittingly enough in his capacity as President of the Royal Astronomical Society, gives an illuminating account of Halley's Magnetic Charts, with some excellent reproductions of these and of other charts by van Bemmelen (1899) for the period 1500-1700. As Professor Chapman remarks, " the accordance shown between the two charts indicates that we have reasonably reliable knowledge of the distribution of the magnetic declination over a large part of the Earth in A.D. 1700 ".

One small but pleasing feature is the inclusion of two Latin verses which appeared on Halley's Chart, *Ad Dominam Reginam* (Queen Anne) and *De Inventore Pyxidis Nauticæ*, with translations by Mrs. Chapman which catch the exact tone of the Augustan age of English verse. T. A. A. B.

Outline of the History of Mathematics. By R. C. Archibald. 5th edition. Pp. 76. 75 cents, postpaid, remittance with order. 1941. (Mathematical Association of America, Oberlin, Ohio)

This pamphlet has become so popular in America that a new edition has had to be brought out barely two years after the fourth edition, and all that could be said about its value and reliability has been further enhanced by the revisions in the new edition. They consist in general of a more careful way of expression (for example, " Pythagoreans " for " Pythagoras "), and of a few additions in the light of recent research. There is, above all, a new index of names ; but the opportunity of giving in this index the life dates of at least the more eminent mathematicians has unfortunately been missed, and the strange discrepancy still persists that such dates are only given for those " minor " scientists who have had to be relegated to the Notes. The attempt

to write names in authentic spelling is, perhaps, doomed to failure after we have once got used to certain forms; but when the younger Bolyai is called "Janos", why not call Desargues "Girard" (as he himself did) or the German-Swiss Euler "Leonhard", and the Bernoullis "Jacob" and "Johann"? Of more general alterations one may note with some amusement the mathematician's reluctant acknowledgment that there is no year 0 B.C.; perhaps a scale A.U.C. or A.M. could please severer critics? More serious (and not accidental in "pages of standing type") seems the abandonment (p. 13), amongst the ideals Greece gave to the world, of the standard of Freedom (while Truth, Beauty and others are still recognised): where, indeed, would Lincoln's country have us search for the "Beacon Lights of Freedom"? The historian of Science may agree to drop titles such as "Freiherr" or "Sir" (Laplace is still "Marquis"); but there is, of course, a sociological note in these data which might induce some reader to undertake further research.

The growing popularity of this *Outline* would seem to necessitate closer scrutiny to avoid the spread of possible misunderstandings, so here are some small points which appear to need further revision:

p. 3, p. 36: James Gregory ought to rank with Wallis and Barrow; *cf.* Turnbull's *Memorial Volume*, 1939; and on p. 73, l. 19, "David" should be inserted.

p. 19, l. –3: to be consistent 22/7 should be written as $3\frac{1}{7}$.

p. 21, l. –7: insert "Fermat" after Viète.

p. 22, l. –7: Dionysodorus? No other reference is made to him, and the non-specialist reader for whom this *Outline* is meant will find it difficult to place him.

p. 27, l. –17: as Koenigsberg is *not* the town in Prussia but a village in South Germany, this fact should be mentioned.

p. 28, l. –10: "Member of Parliament" is a misleading statement for 16th century France; the local "Parlement" of Tours is what is meant (*cf.* Fermat on p. 34). Viète's official capacity as Maître des Requestes is perhaps best described as that of Royal Commissioner.

p. 28, l. –5: for "often" read "consistently".

p. 28, l. –3: after "$\cos\phi$" insert "for any natural number n and wrote it down . . ."

p. 29, l. 4: as a rule, Viète wrote A quad, A cub, etc., the advantage over his predecessors lying in the use of one symbol "A" instead of different symbols for every power.

p. 30, l. 15: not quite exact, as any relation to $1/e$ would require a limit process: Napier's logarithms coincide, to 14 figures, with those referred to in the text.

p. 30, l. 22: it might help if the reason for Napier's choice, the value of the logarithmic sine, was given.

p. 31, l. –10: perhaps better "Believed to have found".

p. 34, l. 10: not correct in this form; *cf.* the cycloid.

p. 34, l. 19: this rather vague statement ought to be corroborated by some reference to the literature on the subject, considering the heat of priority quarrels in the 17th century.

p. 36, l. –10: it seems unfortunate that since the form of Wallis' infinite product for π has been changed in this edition, the best form has not been selected from the various types Wallis gave, *i.e.* that form which appeals most to the modern eye (not ∞/∞)

$$\frac{4}{\pi}=1\times\frac{9}{8}\times\frac{25}{24}\times\frac{49}{48}\times\frac{81}{80}\ldots \textit{ad inf.} \quad (\textit{Alg.}, \text{cap. } 84.)$$

p. 46, l. –6 : but see *Disquis. Arith.* § 365, 366.

And some small misprints :

p. 43, l. 15 : for " 21 " read " 20 " (Frederick died in 1786).
p. 44, l. 3 : for " side " read " sides ".
p. 46, l. –4 : for " four " read " five " ($n=0, 1, 2, 3, 4$).

The text on p. 38, l. –18, and p. 40, l. 8, could be improved by slight modification. A. P.

The Second Year Book of Research and Statistical Methodology Books and Reviews. Edited by OSCAR KRISEN BUROS. Pp. xx, 383. $ 5. 1941. (The Gryphon Press, New Jersey)

In 1937 there was published, as a Rutgers University Bulletin (Studies in Education), *Educational, Psychological, and Personality Tests of 1936*. This was followed by *The 1938 Mental Measurements Year Book*, and from this the section on statistics was republished as a separate volume under the name of *Research and Statistical Methodology Books and Reviews of 1933-1938*. The present volume now follows. It is larger and covers a wider field, both in types of books surveyed and in journals from which reviews are extracted. It comprises 1652 review excerpts from 283 journals (75 British, 181 American) relating to 359 books. Some of these latter were among those dealt with in the earlier volume : there are 234 new books, including those, we are told, published up to 1941, June 1. The books and reviews are in English, and therefore published either, in the main, in Great Britain or in U.S.A. The subject matter of the 359 books is statistics and allied topics : it therefore includes philosophy and logical methodology to historical and bibliographical methods. It ranges from the social function of science and social surveys to business statistics and thesis writing. The history of science is well represented by 21 volumes. Mathematical statistics is covered by 18 volumes and mathematical tables by 17. The origin of the book is evidenced by the 44 volumes under education and psychology as well as the 11 on factor analysis. These last are stimulating reading, for we have here contributions from the chief protagonists in this many-angled fight. The books reviewed include most of those of importance of the time covered. Some of the titles that the reviewer expected are, it may be suggested, not found here because in his enforced removal from libraries he cannot check the dates (of such titles as Political Arithmetic, Leybourne on the falling school population, Dublin and Lotka on 25 years of American Life Experience, Huntingdon on Season of Birth, Vernon on Rating Scales and some of the recent English social surveys), and they may all have been completely dealt with in the earlier edition. The reviewer in this journal of the first volume found that there were more entries from his pen than from anyone else : Dr. Wishart still leads with 22 entries. Another British statistician, M. G. Kendall, follows with 17. There are extracts of 32 reviews from the *Gazette* : it is not often that a reviewer of a second edition can find out what the reviewer of the first said by consulting the text of the book in question.

Though the book reviews are excerpts only, yet the unused portions, judging from the very limited sample we have available, are very small and of the order of 5% of the whole. So we have here practically everything that has been written in the journals surveying all the new material. A humbling experience it is to read these extracts. Your present reviewer tried to assign marks, first to the books reviewed on the basis of the reviews, and then to the reviews, on their value to the intending reader and purchaser. Both attempts failed. What is a good review? What is a good reviewer? What has gone wrong when one writer describes a book (87) as beautifully got up and elegantly

printed (Sarton in *Isis*) and another as inartistically printed (Comrie in this journal), when one reviewer describes a book (313) as good humoured and another speaks of the " note of bitterness " in it, when one reviewer says of the writer of book 341 that " the author has grasped the tail of something which she does not fully comprehend ", another says the work is uniformly scholarly, and a third that the book is imbued with the scientific spirit. The most glaring example perhaps is in connection with the reviews relating to Sherwood Taylor's *Short History of Science* (not reviewed in the *Gazette*), where there are some dozen pairs of mutually contradictory verdicts. These, and similar discrepancies, are not merely on matters of value and of judgment but also of fact. The volume has been prepared with the object, *inter alia*, of improving the quality of book reviews. And there is clear evidence here that certain journals hand reviewing over, as the editors of this book say, to individuals unable to appraise competently or honestly. The worst offenders on this side of the Atlantic appear to be the general and literary weekly periodicals. It seems clear that publishers do themselves no good by sending books for review where they will not get treated by writers with the necessary technical knowledge, intelligence, and disinterestedness. Some reviewers fail for lack of one or other of these qualities ; some by reason of excessively jaundiced or genial temperament ; whilst others are merely exhibitionists displaying through superficial showmanship their ignorance and their distorted rationalisations. The book should thus be very valuable for illustrations in a course on book reviewing. The editors think that it should help students and librarians to select books with greater discrimination : it is undoubtedly a great convenience to have all the relevant matter brought together within one cover, particularly when there has been review, reply, and counterblast. The editor suggests that the volume should be extended to (1) deal with foreign language books, (2) abstract from periodicals and (3) review articles in periodicals. These three features would well make the book even more valuable than perhaps the present objectives, and it is to be hoped that the financial position of statisticians will make it possible. The present work is praiseworthy in its efforts and it has undoubtedly been a source of instruction and benefit to at least one reader.

The book, although large (2 column pages, $10\frac{1}{2}''$ by $7\frac{1}{2}''$) is light to handle and the type is very pleasant. Misprints there are, but they are usually unimportant. Cross references are usually complete, or are easily made with the help of the very useful indices (journals, books, publishers, titles, topics, writers).

FRANK SANDON.

Mathematics for Engineers. By R. W. DULL. 2nd edition. Pp. xviii, 780. 35s. (McGraw-Hill)

This book, which is a second edition, is written by an engineer for engineers. In the preface to the first edition the author states that it is intended for engineers, (1) who want a quick and convenient reference, (2) who have grown somewhat rusty in their mathematics, and (3) who feel the need of a text for the study of mathematics. It might be of considerable use to the first category, but would prove somewhat confusing to the others, particularly the third.

Its 57 chapters take the reader from a set of rules for rapid numerical computation, through algebra, trigonometry, analytical geometry and calculus, finishing with multiple integration. It must not be thought, however, that the whole subject is logically developed. Many results are merely stated without proof; examples of this are the binomial theorem, even for a positive integral index, and the usual formulae of trigonometry including numerous

identities and the formulae for solution of triangles. In other cases proofs which are adequate for the type of reader anticipated are given ; the proof of Taylor's theorem belongs to this class, though it should be noted that while the remainder is specifically included in the statement of the theorem, it is never used in any of the applications. Other proofs, notably the derivation of the exponential and trigonometric series, are definitely unsound, and would have been better dispensed with altogether.

A further blemish on the logical development is the author's habit of using symbols and concepts before they are defined. A few examples of this may be given. On p. 54 there appears the statement

$$\text{“ } (a^2+b^2)=(a+b\sqrt{-1})\,(a-b\sqrt{-1}) \text{ ”,}$$

without any explanation, while the chapter on complex numbers begins on p. 390. Manipulation of radicals is effected by means of fractional indices before the latter are defined, and when this is at last done, no explanations are given. Trigonometric ratios are used before they are defined.

Other examples of confusion are the following. The symbols $\propto$ and ∞ are both used for " infinity " in different parts of the book, and the former is also used for " varies as ". The following is a quotation from p. 69 in a discussion of the graph of the function $y=mx$. " Since all these graphs have constant slopes, they are all straight lines. If the variation was not uniform, the slope would not be uniform and the function would not be of the first degree of x. Therefore, all functions of the first degree are straight lines ". The chapter on implicit quadratic functions contains much relatively heavy algebra, yet in the following chapter it is thought necessary to warn the reader against cancelling terms from the numerator and denominator of a fraction.

Some examples of actual errors have been noted ; some of these, but not all, may be due to misprints. It is stated that $a^{m/n}=a\,.\,a\,.\,a\,.\,a\,\ldots$ etc. to m/n factors, where m/n is clearly intended to be a fraction. The binomial series for $(1\pm x)^n$ is stated to be convergent if $x^2>1$ and the series for $(1\pm x)^{-n}$ is definitely wrong. This last result is stated without any exposition and it is particularly important in such a case that accuracy should be observed. The series given for sin x in powers of $(x-a)$ stops short at the sixth term. There are other similar errors, but the above are sufficient to indicate their type.

There are no examples for the student to work and only comparatively few worked in the text. These do not have, as one might expect, a particularly utilitarian flavour. A special section is given, for example, to " clock problems ". Many more " practical examples " are usually found in textbooks intended for engineers but written by mathematicians.

The book is very well produced and considerable care has been lavished on the figures and on the graphical work, though it must be stated that graphical methods are used in some cases where other methods would be much shorter. For example, to solve the equations $y=3\sin\theta+2\cos\theta$, $y=3\cos\theta+2\sin\theta$, each of the right-hand members is converted to the form $R\sin(\theta+c)$, graphs are drawn and the solution $\theta=n\pi+\frac{1}{4}\pi$ deduced. It is never pointed out that $\tan\theta=1$ gives all that is necessary. Sometimes the graphical method is indispensable, but surely it would be best illustrated by examples which are at least not easier to solve otherwise. A similar instance, though not connected with graphical work, is the solution of $\tan 2\theta=\frac{24}{7}$ by turning it into a quadratic in $\tan\theta$.

There is of course much in the book to which no exception can be taken, but in view of the foregoing remarks the reviewer does not feel that it is one which can be recommended for the use of students. One wonders whether a book with the title " Engineering for Mathematicians " by a mathematician,

would be any more successful, and what reception it would have if reviewed in an engineering journal. N. M. H. LIGHTFOOT.

Algebraic Solid Geometry. By S. L. GREEN. Pp. vii, 132. 6s. 1941. (Cambridge)

It is now well established that the geometry syllabus for Pass, three-subject or Engineering degrees must include the analytical geometry of three dimensions " up to and including the central quadrics ". Experience with students reading for such degrees suggests the usefulness of a textbook, such as the one under review, which contains a large number of graded exercises of a numerical character. Mr. Green's book should be especially welcome to London degree candidates, since most of the exercises are from London University examination papers. But it should also appeal to anybody who wishes for a clear, workaday account of elementary analytical three-dimensional geometry.

The bookwork, for the most part, is neat and thorough. It could perhaps be improved in a very few places. The notion of the angle between two skew lines is a difficult one for most students. The difficulty is not removed by a neat derivation of the $\cos\theta=\cos\alpha\cos\alpha'+\cos\beta\cos\beta'+\cos\nu\cos\nu'$ formula. It is interesting to note that the author omits the $AB\cos\theta$ theorem for the projection of a segment AB on a line skew to it. A suitable proof of this theorem can emphasize the definition of the angle between two skew lines. To turn to another thorny point with beginners, the section on the equations of a line in space, although illustrated by a number of good examples, would still, I think, leave the average student unhappy. As a rule, it is only when a demonstration is given of the extreme ease with which any form of the equations can be translated into any other form that the student feels satisfied.

The $VC \,.\, VD=VA^2$ proof for a harmonic range on p. 47 is surely out of keeping with the rest of the book? Why not use coordinates, taking the origin at the mid-point of AB? Also, is it necessary to know, on p. 89, what *happens* to the roots of $p\lambda^2+2q\lambda+r=0$ as $p\to 0$? Can we not write $\lambda=\mu^{-1}$, and then *put* $p=0$?

As a final adverse criticism I would say that the argument in the section on generators is not satisfying, either algebraically or geometrically. The theorem that through every point of a quadric there pass two generators should be made quite clear.

These very minor blemishes do not affect the total impression one receives of a thoughtful, neat and well-arranged book. Besides the central quadrics the paraboloids are treated. An excellent section emphasises some of the ideas underlying the transformation of axes, and in the chapter on spheres inversion is introduced, and applied to stereographic projection. Within its self-prescribed limits this is a good little book. D. PEDOE.

Elements of Calculus. By A. COHEN. Pp. v, 583. 10s. 6d. 1941. (D. C. Heath, Boston ; Harrap)

As its name implies, this excellent volume contains all the material needed by the student in his first year of a University honours course in mathematics, in so far as the " differential and integral calculus " is concerned.

The first seven chapters, occupying some 200 pages, deal with the idea of differentiation and its geometrical and physical significance. These are followed by six more chapters on integration, and the main part of the book concludes with a discussion of partial differentiation, multiple integrals, infinite series, and the theorems of Taylor and Maclaurin.

In order, as it were, to extend the scope of the subject matter above, the author has been wise enough to provide a number of appendices. These

include a more detailed discussion of limiting processes (so vital to the more intelligent student), the elementary treatment of hyperbolic functions and three-dimensional analytical geometry, and finally a most useful table of formulae and integrals. Numerous examples are worked out in the text, and each chapter is provided with a further selection, to which, unfortunately, no answers are provided.

The book is written in a style which is lucid, logical, and precise; and the writer has been sensible enough to omit proof of those theorems which are outside the scope of the ordinary student, at the same time indicating that he has done so and providing a reference for those who wish to consult it. It seems only fair to point out in passing, that this volume cannot fail to be of use to the teacher, as well as to those being taught; in short, it should prove to be an excellent work of reference for all concerned.

By way of criticism, one feels that the idea of a "differential" might have been used with advantage, at the outset, rather than at the conclusion of the section on differential calculus. A similar objection arises in dealing with integrals; it would seem preferable to define the definite integral as a limit sum [and to deduce the fundamental theorem

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x),$$

and thereby afford a means of calculating it], sooner than to start with the indefinite integral as an antiderivative.

Certain other undesirable, though by no means fundamental, factors arising in the course of the text, are the persistent use of left-handed axes, the laborious and antiquated method used to calculate partial fractions, and finally the definition of a divergent sequence. It must be realised that in any work, however well written, instances, such as these, where the personalities of the author and reviewer clash, are to a certain extent inevitable. J. H. P.

1396. Extracts from *The Western Morning News*, Plymouth.—Letters to the Editor:

13 Trumps.

Sir,—You report that at a whist table at a function at the Royal Hotel, Plymouth, a player was dealt all thirteen trumps. The odds against this happening are 635,013,559,599 to 1. The odds against all four players holding complete suits are 2,235,197,406,895,366,368,301,559,599 to 1. A.

"Impossible to Work Out."

Sir,—The letter from A. stating that the odds against four players holding the complete suits in cards are 2,235,197,406,895,366,368,301,559,599 to one is too ridiculous for words, as it would be humanly impossible for anyone to work out such odds.

In any case, he might have made the last 599 an even 600. B.

Sir,—The odds against all four players at whist holding complete suits are 2,235,197,406,895,366,368,301,559,599 to 1.

These figures are worked out by combinations and permutations, and have been checked by the Royal Statistical Society and other eminent mathematicians. They are undoubtedly correct, and may be accepted as such. A.

[Per Mr. F. W. Kellaway.]

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